Long time behaviors for 3D cubic damped Klein-Gordon equations in inhomogeneous mediums

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Abstract In this paper, we study the asymptotic dynamics of global solutions to damped Klein-Gordon equations in inhomogeneous mediums (KGI). In the defocusing case, we prove for any initial data, the solution is globally define in forward time and it will converge to an equilibrium. In the focusing case, for global solutions, we prove the solutions converge to the superposition of equilibriums among which there exists at most one equilibrium to KGI and the other equilibriums are solutions to stationary nonlinear Klein-Gordon equations. The core ingredients of our proof are the existence of the "concentration-compact attractor" and the gradient system theory.

1 Introduction

The damped Klein-Gordon equation in inhomogeneous mediums is given by

$$\begin{cases} u_{tt} - \Delta u + u + b(x)u + 2\alpha u_t + \lambda |u|^{p-1}u = 0, \\ u(0) = u_0,\ \partial_t u(0) = u_1 \in \mathcal{H}, \end{cases} \tag{1.1}$$

where $\mathcal{H} = H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$, $\alpha \geq 0$, $\lambda \in \mathbb{R}$. In this paper, we consider the three dimensional cubic case, which is of fundamental importance in physics, namely $d = 3, p = 3$. The energy is given by

$$E(f,g) = \int_{\mathbb{R}^3} \left(\frac{1}{2}|\nabla f|^2 + \frac{1}{2}|f|^2 + \frac{1}{2}b(x)\,|f|^2 + \frac{1}{2}|g|^2 + \frac{\lambda}{4}|f|^4 \right) dx.$$

This equation arises in a number of physical settings, for instance the propagation of wave in a nonlinear medium. If the medium has local inhomogeneities, defects or impurities, which appear in the mathematical model as a spatially dependent coefficient in the equation (e.g. localized potential), (1.1) can be used to describe the propagation of the wave. Such perturbations of the original homogeneous (translation invariant) dynamics introduce new modes into the system (impurity modes) which can trap some of the energy and affect the time evolution of the system; see [25, 21, 40].

In this paper, we aim to study the long time behaviors of (1.1). For nontrivial $b(x)$ and $\alpha = 0$, A. Soffer, M. I. Weinstein [35] proved solutions with small data decay to zero, which

implies there exists no quasi-periodic solution in a small neighborhood of zero. When $b(x) = 0$, $\alpha > 0$, E. Feireisl [12] proved the solution either blows up at finite time or globally exists with bounded trajectory for $1 < p < 1 + \min(\frac{d}{d-2}, \frac{4}{d})$ when $d \geq 3$. Meanwhile, he proved for any time sequence, up to a subsequence, the solution decouples into the superposition of equilibriums. N. Burq, G. Raugel, W. Schlag [2] studied the long time behaviors of solutions to nonlinear damped Klein-Gordon equations in the radial case. They proved that radial global solutions will converge to equilibrium points as time goes infinity. All the works mentioned above are related to the so called soliton resolution conjecture in dispersive equations. The (imprecise sense) soliton resolution conjecture states that for "generic" large global solutions, the evolution asymptotically decouples into the superposition of divergent solitons, a free radiation term, and an error term tending to zero as time goes to infinity, for more expressions and history, see A. Soffer [34]. In dissipative systems, the free radiation term vanishes because of the damping effect. Hence for dissipative systems, it is widely conjectured that the solutions will decouple into the superposition of equilibrium points.

There are plenty of works devoted to the verification of the soliton resolution conjecture. For critical dispersive equations especially semilinear wave equations and wave maps, there are a lot of works. T. Duyckaerts, C. Kenig, F. Merle [7] made a breakthrough on this topic by showing the radial solution to three dimensional focusing energy-critical wave equations with bounded trajectory is in fact a superposition of a finite number of rescalings of the ground state plus a radiation term which is asymptotically a free wave. One of the core ingredient of their arguments is the novel tool, called "channels of energy" introduced by [7, 8]. The method developed by them has been applied to many other situations, such as [5, 6, 19, 20] for wave maps, [4, 17] for semilinear wave equations. By a weak version of outer energy inequality, the soliton resolution along a sequence of times was proved by R. Cote, C. Kenig, A. Lawrie and W. Schlag [4] for four dimensional critical wave equations in radial case and by H. Jia, C. Kenig [18] for semilinear wave equations, wave maps.

For subcritical equations, different methods were used to study the long time behaviors of solutions. It is known that the nonlinear Klein-Gordon equation (nonlinear Schrödinger equation) admits a radial positive stationary solution (soliton) with the minimized energy among all the non-zero solutions of

$$-\Delta Q + Q - |Q|^{p-1}Q = 0. \tag{1.2}$$

Besides the ground state, (1.2) also has an infinite number of nodal solutions. (see for instance H. Berestycki, P.L. Lions [1]). Hence it seems that subcritical problems need different techniques.

The dynamics of solutions below and slightly above the ground state for nonlinear Klein-Gordon equations were studied recently. For nonlinear Klein-Gordon equations with initial data

with energy below the ground state, it was proved by S. Ibrahim, N. Masmoudi, K. Nakanishi [16] that the solution either blows up at finite time or scatters to zero. K. Nakanishi, and W. Schlag [26] described the asymptotics of the solutions with energy slightly larger than the ground state. In fact, they proved the trichotomy forward in time: the solution (i) either blows up at finite time (ii) or globally exists and scatters to zero (iii) or globally exists and scatters to the ground state. In the radial setting, the above trichotomy was obtained in K. Nakanishi and W. Schlag [27], followed by K. Nakanishi, W. Schlag [28] in the non-radial case. The main technical ingredient of their papers is the "one pass" theorem which excludes the existence of (almost) homoclinic orbits between the ground state and (almost) heteroclinic orbits connecting ground state Q with $-Q$.

Dynamics for initial data far away from the ground state were analyzed for some models. For nonlinear Klein-Gordon equations, T. Cazenave [3] proved the following dichotomy: solutions either blow up at finite time or are global forward in time and bounded in $\mathcal{H}$, provided $1 < p < \infty$, when $d = 1, 2$ and $1 < p < \frac{d}{d-2}$ if $d \geq 3$. N. Burq, G. Raugel, W. Schlag [2] studied the long time dynamics for damped Klein-Gordon equations in the radial case. By developing some dynamical methods especially invariant manifolds, they proved the ω-limit set of the trajectory is just one single point, hence they showed the dichotomy in forward time (i) the solution either blows up at finite time, (ii) or converges to some equilibrium point. For subcritical nonlinear Schrödinger equations in high diemnsions, T. Tao [37] proved that as time goes to infinity, the global solution with bounded trajectory splits into a radiation term that evolves according to the linear Schrödinger equation, and a remainder which converges in energy space to a "concentration-compact attractor". This attractor consists of the union of almost periodic orbits of the NLS flow in the energy space modulate translations and superpositions. As a corollary, he proved the "petite conjecture" of Soffer [34] in high dimensional non-critical case. In T. Tao [38], he proved as $t \to \infty$, the radial solution to the defocusing Schrödinger equation with a potential splits into a radiation term and a remainder which converges in H^1 to a compact attractor K, which consists of the union of spherically symmetric almost periodic orbits of the NLS flow in H^1. It is novel that K is a global attractor, being independent of the initial energy of the initial data.

In this paper, we aim to study the long time behaviors of damped Klein-Gordon equations in inhomogeneous mediums. The main theorems are as follows:

Theorem 1.1. (*Defocusing Case*) *Let $\alpha > 0$, $\lambda > 0$. Assume that $b(x)$ satisfies the Potential Hypothesis: $b(x)$ is a real-valued function satisfying $|b(x)| + |\nabla b(x)| \lesssim \langle x \rangle^{-\mu}$ for some $\mu > 3$. Then for any initial data $(u_0, u_1) \in \mathcal{H}$, (1.1) is globally well-posed and there exists an equilibrium Q to (1.1) such that*

$$\lim_{t \to \infty} \|(u(t), \partial_t u(t)) - (Q, 0)\|_{\mathcal{H}} = 0.$$

In the focusing case, in order to deal with non-radial data, we need additional assumptions on the potential $b(x)$ to ensure that the small data scattering holds which is important to get the existence of "concentration-compact attractor" in the non-radial case.

Theorem 1.2. *(Focusing Case) Let $\alpha > 0$, $\lambda < 0$, $b(x)$ satisfies the Potential Hypothesis: $b(x)$ is a real-valued function satisfying $|b(x)| + |\nabla b(x)| \lesssim \langle x\rangle^{-\mu}$ for some $\mu > 5$, $\sigma(-\Delta + 1 + b(x)) \subset (0, \infty)$, 0 is neither a resonance nor an eigenvalue of $-\Delta + b(x)$. Then for any initial data $(u_0, u_1) \in \mathcal{H}$:*
(I) The solution $u(x, t)$ to (1.1) with initial data (u_0, u_1) either blows up at finite time or exits globally with bounded trajectory. Moreover, if $u(x, t)$ is global, then for any time sequence $t_n \to \infty$, up to a subsequence, there exist $\{(Q_m, x_{m,n})\}$ such that as $n \to \infty$

$$u(t_n) = \sum_{m=1}^{M} Q(x - x_{m,n}) + o_{H^1}(1),$$

where $(Q_m, x_{m,n})$ satisfies the dichotomy:
(a) $-\Delta Q_m + Q_m + b(x)Q_m + \lambda Q_m^3 = 0, \quad x_{m,n} = 0$;
(b) $-\Delta Q_m + Q_m + \lambda Q_m^3 = 0, \qquad \lim_{n\to\infty} |x_{m,n}| = \infty$.
Meanwhile, we have

$$\lim_{t\to\infty} \|\partial_t u\|_{L^2} = 0.$$

(II) For global solutions $u(x, t)$ to (1.1) with $E(u_0, u_1) < E_{critical}$, where $E_{critical}$ is the energy of the ground state of $-\Delta Q + Q + \lambda Q^3 = 0$, we have for some equilibrium Q to (1.1)

$$\lim_{t\to\infty} \|(u(t), \partial_t u(t)) - (Q, 0)\|_{\mathcal{H}} = 0.$$

Remark 1.1. *If $\lambda < 0$, (u_0, u_1) and $b(x)$ are radial, with the potential hypothesis: $b(x)$ is a real-valued function satisfying $|b(x)| + |\nabla b(x)| \lesssim \langle x\rangle^{-\mu}$ for some $\mu > 3$, we can prove that if the solution is global with bounded trajectory then for some equilibrium Q to (1.1)*

$$\lim_{t\to\infty} \|(u(t), \partial_t u(t)) - (Q, 0)\|_{\mathcal{H}} = 0.$$

It is remarkable that no additional spectrum assumption is needed in the radial case. In fact, the spectrum assumption in Theorem 1.2 is used to prove the small data scattering, which is an essential ingredient in proving energy concentration property namely Lemma 3.6. Lemma 3.6 is important to localize the energy in the spatial space for non-radial data.

Remark 1.2. *On (II) of Theorem 1.2, we give some remarks concerning the rationality of the assumption of $E(u_0, u_1) < E_{critical}$ in (II). When $b(x) < 0$, $\lambda < 0$, P.L. Lions [22] proved (1.1) owns positive stationary solutions say $\mathbf{Q}$ which satisfies $E(\mathbf{Q}, 0) < E_{critical}$. Thus there exists*

(u_0, u_1) (say $(\mathbf{Q}, 0)$) satisfying all the conditions in (II) which leads to a non-zero asymptotic state. Generally, when $b(x) < 0$, $\lambda < 0$, there exist excited states to stationary KGI with energy below $E_{critical}$, thus the asymptotic state in (II) of Theorem 1.2 may also be an excited state.

Remark 1.3. *It seems that there exists an initial data which yields a solution $u(x,t)$ evolving asymptotically as $\bar{Q}(x - x(t)) + Q(x)$, where $\bar{Q}$ is an equilibrium to the nonlinear Klein-Gordon and Q is an equilibrium to (1.1), $\lim\limits_{t\to\infty} |x(t)| = \infty$. At least in the rough sense, if we take $x(t) \sim t^\gamma$ for some $\gamma \in (0,1)$, $\bar{Q}(x - x(t)) + Q(x)$ satisfies (1.1) in the asymptotic sense. We wish to construct such kind of global solutions in our future work.*

Remark 1.4. *Parts of our results can be extended to other p and d by concentration-compact attractor techniques, see Z. Li and L. Zhao [23]. Since the main idea of [23] has been presented in this paper, [23] will not be submitted to any journal for publication.*

We first illustrate the idea of the proof in the simpler defocusing case. The whole proof contains two essential steps, one is to prove the trajectory is pre-compact in some sense in the energy space, the other is to prove all the asymptotic states along different time sequences are the same. In the defocusing case, the first step is reduced to the existence of global attractor which means the maximal compact subset of the energy space invariant under the flow defined by (1.1) which attracts all the trajectory. For $\lambda > 0$, the existence of global attractor to (1.1) was proved by many authors for different p, for instance E. Feireisl [10, 11], M. Prizzi, K.P. Rybakowski [30]. Thus in the defocusing case, we will omit this step in our proof. We remark that the main tool they used in their proof is the tail estimate, which fails in the focusing case.

The focusing case is much more involved. The first step is to prove the trajectory is attracted by concentration-compact attractor (see Definition 4.1). The second is to prove all the asymptotic states along different time sequences are the same. The essential ingredient in the first step is the so-called "concentration-compact attractor". Concentration-compact attractor was introduced in T. Tao [37], it is used to prove the trajectory of the solution excluding the radiation part has a compactness in some sense modulate spatial translations and the superposition. Precisely speaking, it means the trajectory of the solution is attracted by a G-precompact set with J components. The essential ingredient in the second step is Lojasiewicz-Simon inequality in gradient systems. Lojasiewicz-Simon inequality in the gradient system was discovered by L. Simon [36]. It was widely used to study the long time behaviors of dissipative systems on bounded domains, for instance A. Haraux and M.A. Jendoubi [14, 15], P. Rybka, K.H. Hoffmann [33], and see J.K. Hale, G. Raugel [13] for generalizations to gradient-like systems.

Our proof is divided into four parts. In the first step, we prove the trajectory of $u(t)$ is attracted by a G-precompact set with J components, namely the existence of the concentration-compact attractor. The key ingredient in this step is the frequency localization and the spatial localization. In the second step, for any time sequence going to infinity, we prove up to a

subsequence there exist a finite number of asymptotic profiles. Then by applying perturbation theorem, we obtain a nonlinear profile decomposition. Using the damping effect and the fact (1.1) is not translation invariant, we can show all the profiles are equilibriums to (1.1) or the nonlinear Klein-Gordon equation. Finally we prove the convergence for all time. The essential ingredient in this step is the gradient system theory especially Lojasiewicz-Simon inequality which implies all the asymptotic profiles are the same for different time sequences.

Our paper is organized as follows: In Section 2, we recall some preliminaries, such as Strichartz estimates, local well-posedness, perturbation theorem. In Section 3, we prove the frequency localization and spatial localization. In Section 4, we prove the existence of concentration-compact attractor. In Section 5, we finish our proof by using methods from gradient systems.

Notation The Fourier transform on $\mathbb{R}^3$ is defined by

$$\mathcal{F}(f)(\xi) = \tfrac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{-ix\cdot\xi} f(x)\,\mathrm{d}x.$$

We recall the Littlewood-Paley projection defined as follows: Let $\varphi \in C_c^\infty(\mathbb{R}^3)$ be such that $\varphi(\xi) = 1$ for $|\xi| \leq 1$ and $supp\,\varphi(\xi) \subset \{\xi : |\xi| \leq 2\}$. Then we define $\psi_k(\xi) = \varphi\Big(\frac{\xi}{2^k}\Big) - \varphi\Big(\frac{\xi}{2^{k-1}}\Big)$, $\forall\, k \in \mathbb{Z}$. The standard Littlewood-Paley projection operator are defined by $P_{\leq N} f = \phi_0(\frac{\sqrt{-\Delta}}{N}) f$, $P_{\geq N} = I - P_{\leq N}$, $P_k f = \psi_k(\sqrt{-\Delta}) f$. Sometimes for convenience, we denote $P_{<N} f$ by $f_{<N}$. All the constants are denoted by C and they can change from line to line.

2 Preliminaries

In this section, we give the Strichartz estimates, local well-posedness and perturbation theorem. Moreover, we prove that if the solution to (1.1) is global in forward time, then the trajectory of the solution is bounded in $\mathcal{H}$. Consider the linear equation,

$$u_{tt} + 2\alpha u_t - \Delta u + u = G,\ \big(u(t_0, x), u_t(t_0, x)\big) = (u_0, u_1) \in \mathcal{H}, \tag{2.1}$$

then by Duhamel principle,

$$\begin{aligned}
u(t) &= e^{-\alpha(t-t_0)} \left[\cos(t\sqrt{-\Delta + 1 - \alpha^2}) + \alpha \frac{\sin(t\sqrt{-\Delta+1-\alpha^2})}{\sqrt{-\Delta+1-\alpha^2}} \right] u_0 \\
&\quad + e^{-\alpha(t-t_0)} \frac{\sin(t\sqrt{-\Delta+1-\alpha^2})}{\sqrt{-\Delta+1-\alpha^2}} u_1 + \int_{t_0}^t \frac{\sin((t-s)\sqrt{-\Delta+1-\alpha^2})}{\sqrt{-\Delta+1-\alpha^2}} e^{-\alpha(t-s)} G(s) ds \\
&\triangleq S_{1,\alpha}(t) u_0 + S_{2,\alpha}(t) u_1 + \int_{t_0}^t S_{2,\alpha}(t-s) G(s) ds.
\end{aligned}$$

Define $S_L(t)(u_0, u_1) = S_{1,\alpha} u_0 + S_{2,\alpha} u_1$.

The Strichartz estimates and energy bounds are proved by [2].

Lemma 2.1. *Let $t_0 > T > 0$, u be a solution to (2.1) on $[t_0 - T, t_0 + T] \times \mathbb{R}^3$. Let $I \equiv [t_0 - T, t_0 + T]$, then*

$$\sup_{t \in I} \|(u(t), \partial_t u(t))\|_{\mathcal{H}} + \|u\|_{L_t^4(I;L_x^{12})} + \|u\|_{L_t^\infty(I;L_x^6)} + \|u\|_{L_t^5(I;L_x^{10})} \lesssim e^{T\alpha} \left[\|(u_0, u_1)\|_{\mathcal{H}} + \int_I \|G(s)\|_{L_x^2} ds \right].$$

There exists some $\beta > 0$ depending on $\alpha > 0$ such that for $t > t_0$, we have the energy bounds with decay

$$\|(u(t), \partial_t u(t))\|_{\mathcal{H}} \lesssim e^{-\beta(t-t_0)} \|(u(t_0).\partial_t u(t_0))\|_{\mathcal{H}} + \int_{t_0}^t e^{-\beta(t-s)} \|G(s)\|_{L^2} ds.$$

As a corollary of Strichartz estimates, we have the perturbation theorem.

Lemma 2.2. *Let $M > 0$, $3 < Q < \infty$, there exists $\varepsilon_0 = \varepsilon_0(M)$ satisfying the following: Let $I \subset \mathbb{R}^+$ be a finite interval containing t_0, $\tilde{u}$ is defined on $I \times \mathbb{R}^3$, and satisfies*

$$\sup_{t \in I} \|(\tilde{u}, \partial_t \tilde{u})(t)\|_{\mathcal{H}} \leq M.$$

Suppose that v is a solution to (1.1) with initial data $(v(t_0), \partial_t v(t_0))$ at time t_0. Let $\varepsilon \in (0, \varepsilon_0)$, suppose that

$$\partial_{tt} \tilde{u} + 2\alpha \partial_t \tilde{u} - \Delta \tilde{u} + \tilde{u} + b(x)\tilde{u} + \lambda |\tilde{u}|^2 \tilde{u} = e,$$

$$\|e\|_{L_t^1(I;L_x^2)} + \|S_{1,\alpha}(t - t_0)(\tilde{u} - v)(t_0)\|_{L_t^Q(I;L_x^6)} + \|S_{2,\alpha}(t - t_0)(\partial_t \tilde{u} - \partial_t v)(t_0)\|_{L_t^Q(I;L_x^6)} \leq \varepsilon.$$

Then

$$\|\tilde{u} - v - S_L(t - t_0)(\tilde{u} - v)(t_0)\|_{L_t^\infty(I;\mathcal{H})} + \|\tilde{u} - v\|_{L_t^Q(I;L_x^6)} \leq C(M, |I|)\varepsilon.$$

By contraction mapping principle, we can prove the following local well-posedness theorem:

Proposition 2.3. *For $(u_0, u_1) \in \mathcal{H}$, there exists $T > 0$ such that (1.1) is well-defined in $[0, T)$, with T depending on $\|(u_0, u_1)\|_{\mathcal{H}}$.*

Proposition 2.4. *Suppose that $b(x)$ is a potential in Theorem 1.2. If $\lambda < 0$, $\|(u_0, u_1)\|_{\mathcal{H}} < \epsilon$ with ϵ sufficiently small, then there exits $\gamma > 0$ such that*

$$\|(u(t), \partial_t u(t))\|_{\mathcal{H}} \leq C e^{-\gamma t} \|(u_0, u_1)\|_{\mathcal{H}}.$$

Proof. By Birmann-Schwinger principle there exist a finite number of bound states of $-\Delta + b(x) + 1$ say $\xi_1, ..., \xi_N$. Denote $H = -\Delta + b(x) + 1$, the spectrum assumption in Theorem 1.2 shows $H\xi_i = \nu_i \xi_i$, for $\nu_i > 0$, $i = 1, ..., N$. Using the boundedness of the wave operator proved

by K. Yajima [39], it is easy to extend the energy bounds in Lemma 2.2 of [2] to $-\Delta + b(x) + 1$ case, namely if v satisies

$$\partial_t^2 v + 2\alpha\partial_t v - \Delta v + v + b(x)v = G, \quad (v, v_t)|_{t=0} = (v_0, v_1),$$

then we have

$$\|(P_c(H)v, P_c(H)v_t)\|_{\mathcal{H}} \lesssim e^{-\beta t}\|(u_0, u_1)\|_{\mathcal{H}} + \int_0^t e^{-\beta(t-s)}\|P_c(H)G(s)\|_{L^2}ds, \tag{2.2}$$

where $P_c(H)$ is the projection to the continuous part of H. Applying $P_c(H)$ to (1.1), we obtain from (2.2) that for some $C^*, \beta > 0$

$$\|(P_c(H)u, P_c(H)\partial_t u)\|_{\mathcal{H}} \le C^* e^{-\beta t}\|(u_0, u_1)\|_{\mathcal{H}} + C^* \int_0^t e^{-\beta(t-s)} \|u\|_{L^6}^3 \, ds. \tag{2.3}$$

Let $\omega_i(t) = \langle u(t), \xi_i \rangle$, $i = 1, ..., N$. Then ω_i satisfies

$$\partial_t^2 \omega_i(t) + 2\alpha\partial_t\omega_i + \nu_i\omega_i = \lambda \left\langle u^3, \xi_i \right\rangle.$$

By Duhamel principle, we deduce

$$\begin{aligned}
\omega_i(t) = e^{-\alpha t}&\left[\frac{1}{2}\left(e^{\sqrt{\alpha^2-\nu_i}t} + e^{-\sqrt{\alpha^2-\nu_i}t}\right)\omega_i(0) + \frac{1}{2\sqrt{\alpha^2-\nu_i}}\left(e^{\sqrt{\alpha^2-\nu_i}t} - e^{-\sqrt{\alpha^2-\nu_i}t}\right)\partial_t\omega_i(0)\right] \\
&+ \lambda e^{-\alpha t}\int_0^t \frac{1}{2\sqrt{\alpha^2-\nu_i}}\left(e^{\sqrt{\alpha^2-\nu_i}(t-s)} - e^{-\sqrt{\alpha^2-\nu_i}(t-s)}\right)e^{\alpha s}\left\langle u^3(s), \xi_i\right\rangle ds,
\end{aligned} \tag{2.4}$$

where when $v_i = \alpha^2$, we make the convention that

$$\frac{1}{2\sqrt{\alpha^2-\nu_i}}\left(e^{\sqrt{\alpha^2-\nu_i}t} - e^{-\sqrt{\alpha^2-\nu_i}t}\right) = t,$$

and $\Im\sqrt{a} \ge 0$. Splitting u into the discrete parts and the continuous part gives

$$u(t) = \sum_{i=1}^N \omega_i(t)\xi_i + P_c(H)u(t).$$

Suppose that $\|(u_0, u_1)\|_{\mathcal{H}} \le \varepsilon_1$ for some $\varepsilon_1 > 0$ sufficiently small to be determined. Let $\gamma > 0$ be some sufficiently small constant to be determined later, for a time interval $I = [0, h)$, we define

$$\Phi_I = \sup_{t\in I, i\in\{1,..,d\}} e^{\gamma t}\left|\omega_i(t)\right| + \sup_{t\in I} e^{\gamma t}\left\|(P_c(H)u(t), \partial_t P_c(H)u(t))\right\|_{\mathcal{H}}.$$

Assume that $\Phi_I < (100 + C^*)\varepsilon_1$, if we can prove $\Phi_I \le 2C^*\varepsilon_1$, then by the continuity method, we can obtain $I = [0, \infty)$, thus finishing the proof of Lemma 2.4. Since $\nu_i > 0$, for $\gamma <$

$\min_i \left\{\alpha - \sqrt{\alpha^2-\nu_i}, \frac{1}{2}\beta\right\}$, (2.3) and (2.4) yield

$$\Phi_I \leq C^*\varepsilon_1 + C^*\Phi_I^3 \int_0^\infty e^{-(\frac{1}{2}\beta+3\gamma)(t-s)} ds.$$

Choosing ε_1 to be sufficiently small, we can conclude $\Phi_I \leq 2C^*\varepsilon_1$. $\square$

The following dichotomy lemma shows the global solution to (1.1) has bounded trajectory. We use the convexity arguments in [3, 2]. The proof is standard, for reader's convenience, the detail is given in Appendix A.

Lemma 2.5. *For $b(x)$ in Theorem 1.2, $\lambda < 0$, any initial data $(u_0, u_1) \in \mathcal{H}$ we have*
(i) either $u(x,t)$ blows up at finite time,
(ii) or $u(x,t)$ exists globally and

$$\sup_{t\in[0,\infty)} \|(u, \partial_t u)(t)\|_{\mathcal{H}} \leq C\|(u_0, u_1)\|_{\mathcal{H}}.$$

For stationary solutions, we have the following lemma.

Lemma 2.6. *Let $b(x)$ be a potential in Theorem 1.2, $\lambda \in \mathbb{R}$. If $Q \in H^1$ is a stationary solution to (1.1), then $Q \in H^2 \cap C^2$, and for $|k| \leq 1$, $\partial_x^k Q(x)$ is of exponential decay. Moreover, there exists $\kappa > 0$ such that every nonzero equilibrium Q to (1.1) satisfies $E(Q, 0) > \kappa$.*

Proof. $Q \in H^2 \cap C^2$ is standard by elliptic regularity theorems. Theorem 4.2 in P. J. Rabier and C. A. Stuart [29] implies if $Q \in W^{1,p} \cap W^{2,3}_{loc}$ for some $p > 3$, then $\partial_x^k Q(x)$ decays exponentially for $|k| \leq 1$. Thus by Sobolev embedding it suffices to prove $Q \in H^3$. Since $(-\Delta + 1)Q_x = -b(x)Q_x - \partial_x b(x)Q - \lambda 3Q^2 Q_x$, $Q \in H^2$, Sobolev embedding gives $(-\Delta + 1)Q_x \in L^2$, which shows $Q \in H^3$. Now, we turn to prove $E(Q, 0) > \kappa$ by contradiction. First, we observe that

$$E(Q, 0) = \frac{1}{4} \langle HQ, Q \rangle,$$

which combined with $\sigma(-\Delta + 1 + b(x)) = (0, \infty)$ gives

$$E(Q, 0) \geq 0.$$

Therefore, if Lemma 2.6 fails, then there exist $Q_n \neq 0$ such that

$$-\Delta Q_n + Q_n + b(x)Q_n + \lambda Q_n^3 = 0, \tag{2.5}$$

and

$$\lim_{n\to\infty} \frac{1}{2} \langle HQ_n, Q_n \rangle + \frac{\lambda}{4}\|Q_n\|_4^4 = 0. \tag{2.6}$$

Thus (2.5) and (2.6) yield

$$\lim_{n\to\infty} \langle HQ_n, Q_n\rangle = 0. \tag{2.7}$$

By Birmann-Schwinger principle and the assumptions on $b(x)$, there exists $\sigma > 0$ such that for any $f \in D(H) = H^2$

$$\langle Hf, f\rangle \geq \sigma \|f\|_2^2 . \tag{2.8}$$

We claim in fact

$$\langle Hf, f\rangle \gtrsim \|f\|_{H^1}^2 . \tag{2.9}$$

Suppose that (2.9) fails, then there exists $f_n \in H^2$ such that

$$\langle Hf_n, f_n\rangle \leq \frac{1}{n} \|f_n\|_{H^1}^2 . \tag{2.10}$$

Without loss of generality, we assume $\|f_n\|_{H^1} = 1$. From (2.8), we have $\|f_n\|_2 \to 0$, thus

$$\lim_{n\to\infty} \int_{R^3} b(x)|f_n|^2 dx \to 0,$$

which combined with (2.10) implies

$$\lim_{n\to\infty} \|f_n\|_{H^1} = 0,$$

thus contradicting with $\|f_n\|_{H^1} = 1$. Hence, (2.9) follows. Thus (2.7) reveals

$$\lim_{n\to\infty} \|Q_n\|_{H^1} = 0. \tag{2.11}$$

We have from Gagliardo-Nirenberg inequality that

$$\|Q_n\|_4^4 \leq \|\nabla Q_n\|_2^3 \|Q_n\|_2.$$

Thus by (2.9), (2.11), for n sufficiently large, we deduce

$$0 = \langle HQ_n, Q_n\rangle + \lambda \|Q_n\|_4^4 \geq c \|Q_n\|_{H^1}^2 + \lambda \|Q_n\|_{H^1}^4 \geq \frac{c}{2} \|Q_n\|_{H^1}^2 , \tag{2.12}$$

which contradicts with $Q_n \neq 0$. $\square$

Remark 2.1. *If $b(x) < 0$, by the classic theorem of P.L. Lions [22], (1.1) owns positive stationary solutions. Whether the stationary solutions are hyperbolic or not is not important in*

our proof. In fact, our methods can deal with both non-hyperbolic and hyperbolic case.

3 Frequency localization and Spatial localization

Since we focus on bounded solution throughout the paper, we assume

$$\sup_{t\in[0,\infty)} \|(u,\partial_t u)(t)\|_{\mathcal{H}} \leq E.$$

Then energy equality implies

Corollary 3.1. *If the solution to (1.1) is global, then we have*

$$\int_0^\infty \|\partial_s u(s)\|_{L_x^2}^2 ds < \infty.$$

In the first step, we prove the localization of frequency, namely

Lemma 3.2. *For any $\mu_0 > 0$ there exists $c(\mu_0) > 0$ depending on E such that*

$$\limsup_{t\to\infty} \|P_{\geq \frac{1}{c(\mu_0)}} u(t)\|_{H^1} \leq \mu_0, \tag{3.1}$$

$$\limsup_{t\to\infty} \|P_{\geq \frac{1}{c(\mu_0)}} \partial_t u(t)\|_{L^2} \leq \mu_0. \tag{3.2}$$

Proof. Let $h(u) = \lambda u^3 - b(x)u$. Choose $\delta > 0$ to be some sufficiently large constant such that $e^{-\alpha\delta} \ll \mu_0$. Fix $t_0 > \delta$, consider the interval $I \equiv [t_0 - \delta, t_0]$. By Duhamel principle, for $t \in I$, we have

$$u(t) = S_{1,\alpha}(t - t_0 + \delta)u(t_0 - \delta) + S_{2,\alpha}(t - t_0 + \delta)u(t_0 - \delta) + \int_{t_0-\delta}^{t} S_{2,\alpha}(t-s)h(u(s))ds.$$

Since

$$\left\|S_{1,\alpha}P_{\geq\frac{1}{\mu}}u_0\right\|_{H^1} \leq e^{-\beta t}\|u_0\|_{H^1}, \quad \left\|S_{2,\alpha}P_{\geq\mu^{-1}}u_1\right\|_{H^1} \leq e^{-\beta t}\|u_1\|_{L^2},$$

we have

$$\left\|P_{\geq\mu^{-1}}u(t_0)\right\|_{H^1} \leq C(E)e^{-\beta\delta} + \int_{t_0-\delta}^{t_0} e^{-\beta(t-s)}\left\|P_{\geq\mu^{-1}}h(u(s))\right\|_2 ds.$$

Hence it suffices to bound $\left\|P_{\geq\mu^{-1}}h(u)\right\|_2$. Fix ε sufficiently small, divide I into subintervals $I_1, I_2, ..., I_n$, such that $|I_j| \sim \varepsilon$, then $n \sim \frac{\delta}{\varepsilon}$. For $2 < R < 4$, Hölder's inequality and Strichartz estimates give

$$\begin{aligned}\|u(t)\|_{L^R(I_j;L_x^{12})} &\leq |I_j|^{\frac{1}{R}-\frac{1}{4}}\|u(t)\|_{L_t^4(I_j;L_x^{12})}\\ &\lesssim |I_j|^{\frac{1}{R}-\frac{1}{4}}E + |I_j|^{\frac{1}{R}-\frac{1}{4}}\|h(u(s))\|_{L_t^1(I_j;L_x^2)}.\end{aligned}$$

By Hölder's inequality and Sobolev embedding, we obtain

$$\|h(u(s))\|_{L^1_t(I_j;L^2_x)} \le \int_{I_j} \big\|u^2\big\|_{L^6_x}\|u\|_{L^3_x}dt+\int_{I_j} \|u\|_{L^{12}_x}dt \le C(E)|I_j|^{-\frac{2}{R}+1}\big(\|u\|^2_{L^R_t(I_j;L^{12}_x)}+\|u\|_{L^R_t(I_j;L^{12}_x)}\big).$$

Thus

$$\|u(t)\|_{L^R_t(I_j;L^{12}_x)} \le C(E)\varepsilon^{\frac{1}{R}-\frac{1}{4}}\big(1+|\varepsilon|^{-\frac{2}{R}+1}\big(\|u\|^2_{L^R_t(I_j;L^{12}_x)}+\|u\|_{L^R_t(I_j;L^{12}_x)}\big)\big).$$

By continuity method, we have

$$\|u(t)\|_{L^R_t(I_j;L^{12}_x)} \le 8C(E)\varepsilon^{\frac{1}{R}-\frac{1}{4}}.$$

Summing up all the intervals, we get

$$\|u(t)\|_{L^R_t(I;L^{12}_x)} \lesssim C(\delta,E), \tag{3.3}$$

where $C(E,\delta)$ is independent of t.

Split u into $u = P_{\le\mu^{-1}}u + P_{\ge\mu^{-1}}u$, let $g(u)=u^3$, then

$$g(u) = g\big(P_{\le\mu^{-1}}u\big) + \big(P_{\ge\mu^{-1}}u\big)O(u^2).$$

Bernstein's inequlity and Sobolev embedding imply

$$\begin{aligned}
\big\|P_{\ge\mu^{-1}}g(u)\big\|_2 &\le \big\|P_{\ge\mu^{-1}}g(P_{\le\mu^{-1}}u)\big\|_2 + \big\|P_{\ge\mu^{-1}}\left(P_{\ge\mu^{-1}}uO(u^2)\right)\big\|_2 \\
&\lesssim \mu\big\|\nabla g(P_{\le\mu^{-1}}u)\big\|_2 + \big\|P_{\ge\mu^{-1}}uO(u^2)\big\|_2 \\
&\lesssim \mu\Big\|\big(\nabla P_{\le\mu^{-1}}u\big)\big|P_{\le\mu^{-1}}u\big|^2\Big\|_2 + \big\|P_{\ge\mu^{-1}}u\big\|_3\big\|u^2\big\|_6 \\
&\lesssim \mu\big\|\nabla P_{\le\mu^{-1}}u\big\|_2\,\big\|P_{\le\mu^{-1}}u\big\|^2_\infty + \big\|P_{\ge\mu^{-1}}u\big\|_3\big\|u^2\big\|_6 \\
&\lesssim \mu\|u\|_{H^1}\,\big\|P_{\le\mu^{-1}}u\big\|^2_\infty + \big\|P_{\ge\mu^{-1}}u\big\|_3\,\|u\|^2_{12}.
\end{aligned}$$

Applying Bernstein's inequality, we get

$$\Big\|\big|P_{\le\mu^{-1}}u\big|^2\Big\|_\infty \le \mu^{-\frac{1}{2}}\,\|u\|^2_{12},$$

which combined with Hölder's inequality and (3.3) gives

$$\begin{aligned}
&\int_{t_0-\delta}^{t_0} e^{-\beta(t-s)}\big\|P_{\ge\mu^{-1}}g\left(P_{\le\mu^{-1}}u(s)\right)\big\|_{L^2_x}ds \\
&\le C(E)\mu^{\frac{1}{2}}\int_{t_0-\delta}^{t_0} e^{-\beta(t-s)}\,\|u\|^2_{L^{12}_x}\,ds \\
&\le C(E)\mu^{\frac{1}{2}}\,\|u\|^2_{L^R_t(I;L^{12}_x)}
\end{aligned}$$

$$\leq C(E,\delta)\mu^{\frac{1}{2}}. \tag{3.4}$$

This bound is acceptable. By Littlewood-Paley, we have

$$\begin{aligned}
\left\|P_{\geq\mu^{-1}}u\right\|_3 &\leq \left(\sum_{2^j\geq\frac{1}{4}\mu^{-1}}^{\infty}\|P_ju\|_3^2\right)^{1/2} \lesssim \left(\sum_{2^j\geq\frac{1}{4}\mu^{-1}}^{\infty}2^{-j}2^{2j}\|P_ju\|_2^2\right)^{1/2} \\
&\lesssim \mu^{\frac{1}{2}}\left(\sum_{2^j\geq\frac{1}{4}\mu^{-1}}^{\infty}2^{2j}\|P_ju\|_2^2\right)^{1/2},
\end{aligned}$$

thus similar arguments reveal

$$\begin{aligned}
&\int_{t_0-\delta}^{t_0}e^{-\beta(t-s)}\left\|P_{\geq\mu^{-1}}\left(P_{\geq\mu^{-1}}uO(u^2)\right)\right\|_2 ds \\
&\leq C(E,\delta)\mu^{\frac{1}{2}}. \tag{3.5}
\end{aligned}$$

We obtain from Bernstein's inequality that

$$\begin{aligned}
&\int_{t_0-\delta}^{t_0}e^{-\beta(t-s)}\left\|P_{\geq\mu^{-1}}\big(b(x)u(x)\big)\right\|_2 ds \\
&\lesssim \int_{t_0-\delta}^{t_0}e^{-\beta(t-s)}\mu\,\|b(x)u(x)\|_{H^1}\,ds \\
&\lesssim C(E,\delta)\mu. \tag{3.6}
\end{aligned}$$

Therefore, (3.1) follows from (3.4), (3.5), (3.6). Next, we prove (3.2). From Duhamel principle, we have

$$\begin{aligned}
\partial_t u(t) = &-\alpha u(t) + e^{-\alpha\delta}\left[-\sqrt{\Delta+1-\alpha^2}\sin\left(t\sqrt{\Delta+1-\alpha^2}\right)+\alpha\cos\left(t\sqrt{\Delta+1-\alpha^2}\right)\right]u(t-\delta) \\
&+e^{-\alpha\delta}\cos\left(t\sqrt{\Delta+1-\alpha^2}\right)\partial_t u(t-\delta)+\int_{t-\delta}^{t}\cos\left((t-s)\sqrt{\Delta+1-\alpha^2}\right)e^{-\alpha(t-s)}h\big(u(s)\big)ds.
\end{aligned}$$

For $\mu_1 \ll \mu_0$, (3.1) implies that there exist $\eta > 0$ and $T_0 > 0$ such that

$$\|P_{\geq\eta^{-1}}u(t)\|_{H^1} < \mu_1,$$

for $t > T_0$. Taking δ large such that $e^{-\alpha\delta} < \mu_1$, then for $t > T_0 + \delta$, it suffices to prove

$$\left\|P_{\geq\eta^{-1}}h(u(t))\right\|_2 \leq \eta^{\lambda},$$

for some $\lambda > 0$. The rest of the proof of (3.2) is the same as (3.1). Therefore, we finish our

proof. □

Now, we prove the spatial localization, namely the following proposition:

Proposition 3.3. *Let u be a global solution to (1.1) with $\mathcal{H}$ norm at most $E > 0$. Then there exit $J = J(E)$ depending only on E, and functions $x_1(t), ..., x_J(t) : \mathbb{R}^+ \to \mathbb{R}^3$, such that for any $\mu > 0$ there exit $\eta = \eta(E, \mu) > 0$ such that*

$$\limsup_{t\to\infty} \int_{dist(x,\{0,x_1(t),...,x_J(t)\})>\eta^{-1}} |\nabla u|^2 + |u|^2 + |\partial_t u|^2 \le \mu.$$

Before proving Proposition 3.3, we first prove a weaker proposition:

Proposition 3.4. *Let u be a global solution to (1.1) with $\mathcal{H}$ norm at most $E > 0$. Then for $\mu_0 > 0$, there exits $J = J(E, \mu_0)$ and functions $\tilde{x}_1(t), ..., \tilde{x}_J(t) : \mathbb{R}^+ \to \mathbb{R}^3$, and $\eta = \eta(E, \mu_0) > 0$ such that*

$$\limsup_{t\to\infty} \int_{dist(x,\{0,\tilde{x}_1(t),...,\tilde{x}_J(t)\})>\eta^{-1}} |u|^2 \le \mu_0.$$

Proof. The whole proof is divided into five parts. Fix $E > 0$ and μ_0, choose parameters $\mu_0 \gg \mu_1 \gg \mu_2 \gg \mu_3 \gg \mu_4 > 0$.

Step One. Select a "good" time sequence. For any $t_0 > T_0$, consider the time interval $[t_0 - \mu_1^{-1}, t_0 + \mu_1^{-1}]$. Since

$$\lim_{t_0\to\infty} \int_{t_0-\mu_1^{-1}}^{t_0+\mu_1^{-1}} \|\partial_t u(s)\|_2^2 ds = 0,$$

there exists T_1 sufficiently large such that for $t_0 > T_1$,

$$\int_{t_0-\mu_1^{-1}}^{t_0+\mu_1^{-1}} \|\partial_t u(s)\|_2^2 ds \le \mu_2^2.$$

Thus there exits good time $t_* \in [t_0 - \mu_1^{-1}, t_0 + \mu_1^{-1}]$, such that

$$\|\partial_t u(t_*)\|_2 \le \mu_2^2. \tag{3.7}$$

Step Two. L_x^∞ spatial localization at fixed time. From Lemma 3.1, for any $\mu_2 > 0$ there exists $c(\mu_2) > 0$, such that for $T > T_0$,

$$\|u_{>c(\mu_2)^{-1}}\|_{H^1} \le \mu_2^2. \tag{3.8}$$

As step one, we fix time $t > T_1$. Now we claim there exist $J(E, \mu_2, \mu_3)$ and $x_1(t), ..., x_J(t) : \mathbb{R}^+ \to \mathbb{R}^3$, such that

$$|u_{<c(\mu_2)^{-1}}(t, x)| < \mu_3, \text{ whenever } dist(x, \{x_1(t), ..., x_J(t)\}) \ge 2\mu_3^{-1}. \tag{3.9}$$

Indeed, let $x_1(t), ..., x_J(t)$ be a maximal $2\mu_3^{-1}$-separated set of points in $\mathbb{R}^3$ such that

$$|u_{<c(\mu_2)^{-1}}(t, x_j(t))| \geq \mu_3 \ \text{ for all } \ 1 \leq j \leq J(t).$$

It is easy to verify

$$|u_{<c(\mu_2)^{-1}}(t, x_j(t))| \lesssim \big(c(\mu_2)\big)^{-3/2} \int_{|x - x_j(t)| \leq \mu_3^{-1}} |u|^2 dx + \mu_3^3 \|u\|_2.$$

Then we have

$$\mu_3 \leq |u_{<c(\mu_2)^{-1}}(t, x_j(t))| \lesssim \big(c(\mu_2)\big)^{-3/2} \int_{|x - x_j(t)| \leq \mu_3^{-1}} |u|^2 dx.$$

Since $x_j(t)$ are $2\mu_3^{-1}$-separated, thus J is finite depending on μ_2, μ_3. By the maximal property of the set $\{x_1, ..., x_J\}$, we conclude

$$|u_{<c(\mu_2)^{-1}}(t, x_j(t))| < \mu_3, \ \text{ whenever } \ dist(x, \{x_1, ..., x_J\}) \geq 2\mu_3^{-1}.$$

Step Three. L_x^∞ spatial localization on an interval centered at the good time. For $t > T_1$, consider good time t_* in $[t - \mu_1^{-1}, t + \mu_1^{-1}]$. Then $[t - \mu_1^{-1}, t + \mu_1^{-1}] \subset [t_* - 4\mu_1^{-1}, t_* + 4\mu_1^{-1}] \equiv I$. Define the distance function $D(x) = dist(x, \{0, x_1(t_*), x_2(t_*), ..., x_J(t_*)\})$. Let $\chi : \mathbb{R}^3 \to R^+$ be a smooth cutoff function which equals 1 for $D(x) \leq 2\mu_3^{-1}$, vanishes for $D(x) \geq 3\mu_3^{-1}$, and $\nabla^k \chi = O_k(\mu_3^k)$ for $k \geq 0$. Then we have

Claim 1. $\|S_{1,\alpha}[(1-\chi)u(t_*)]\|_{L_t^Q(I;L_x^6)} + \|S_{2,\alpha}[(1-\chi)u(t_*)]\|_{L_t^Q(I;L_x^6)} \lesssim_{\mu_1} \mu_2^2$.

By Strichartz estimates,

$$\|S_{2,\alpha}[(1-\chi)u(t_*)]\|_{L_t^Q(I:L_x^6)} \leq Ce^{C\mu_1^{-1}} \|\partial_t u(t_*)\|_2,$$

which combined with (3.7) yields the desired bounds for $S_{2,\alpha}$. Since high frequency is small by (3.8), it suffices to prove

$$\big\|S_{1,\alpha}[(1-\chi)P_{\leq c(\mu_2)^{-1}} u(t_*)]\big\|_{L_t^Q(I;L_x^6)} \lesssim_{\mu_1} {\mu_2}^2.$$

From the rapid decay of the convolution kernel of $P_{<c(\mu_2)^{-1}}$ and the support of $1 - \chi$, we see that $(1-\chi)P_{<c(\mu_2)^{-1}}(1_{D<\mu_3^{-1}} P_{<c(\mu_2)^{-1}} u)$ can be absorbed by μ_2^2, it suffices to prove

$$\Big\|S_{1,\alpha}\Big[(1-\chi)P_{<c(\mu_2)^{-1}}\Big(1_{D>\mu_3^{-1}} P_{<c(\mu_2)^{-1}} u(t_*)\Big)\Big]\Big\|_{L_t^Q(I;L_x^6)} \leq \mu_2^2. \tag{3.10}$$

Stationary phase shows that the operator $S_{1,\alpha}(1-\chi)P_{<c(\mu_2)^{-1}}$ has an operator norm of $C_1(\mu_2)$

on L^6. Thanks to (3.9), for some $\delta > 0$, we have

$$
\begin{aligned}
&\Big\| S_{1,\alpha}\left[(1-\chi)P_{<c(\mu_2)^{-1}}\left(1_{D>\mu_3^{-1}}P_{<c(\mu_2)^{-1}}u(t_*)\right)\right]\Big\|_{L_t^Q(I;L_x^6)} \\
&\le C(\mu_2)\Big\|1_{D>\mu_3^{-1}}P_{<c(\mu_2)^{-1}}u(t_*)\Big\|_{L_t^Q(I;L_x^6)} \\
&\le C(\mu_2)\Big\|1_{D>\mu_3^{-1}}P_{<c(\mu_2)^{-1}}u(t_*)\Big\|_{L_t^Q(I;L_x^\infty)}^{\delta}\Big\|1_{D>\mu_3^{-1}}P_{<c(\mu_2)^{-1}}u(t_*)\Big\|_{L_t^Q(I;L_x^2)}^{1-\delta} \\
&\le C(\mu_2,\mu_1)\mu_3^{\delta}\lesssim \mu_2^2.
\end{aligned}
$$

Claim 2. $\|1_{D>\mu_4^{-2}}u\|_{L_t^Q(I:L_x^6)} \lesssim_{\mu_1} \mu_2$.
This claim can be proved by perturbation theorem and Strichartz estimates. Indeed, let v be a solution to (1.1) on I with initial data $v(t_*) = \chi u(t_*)$, $\partial_t v(t_*) = \partial_t u(t_*)$. Then by perturbation theorem, Claim 1, (3.7), we have

$$\|u-v\|_{L_t^Q(I;L_x^6)} \lesssim_{\mu_1} \mu_2^2.$$

It suffices to prove

$$\|1_{D>\mu_4^{-2}}v\|_{L_t^Q(I;L_x^6)} \lesssim_{\mu_1} \mu_2^2. \tag{3.11}$$

Choosing another weight function $W : \mathbb{R}^3 \to \mathbb{R}^+$ comparable to $1+\mu_4 D$ which obeys the bounds $\nabla W, \nabla^2 W = O(\mu_4)$. Since $W\chi = O(1)$, we have

$$\|Wv(t_*)\|_2 \lesssim 1.$$

Since v solves (1), we have

$$\partial_{tt}(Wv) + 2\alpha\partial_t(Wv) - \Delta(Wv) + Wv = W|v|^2 v + O(\mu_4\,|v|) + O(\mu_4\,|\nabla v|) + b(x)Wv.$$

Strichartz estimates imply

$$\|Wv\|_{L_t^5(I';L_x^{10})} + \big\|\big(Wv,\partial_t(Wv)\big)\big\|_{L_t^\infty(I';\mathcal{H})} \lesssim \big\|(Wv(t'),\partial_t Wv(t'))\big\|_{\mathcal{H}} + \Big\|W|v|^3 + b(x)Wv\Big\|_{L_t^1(I';L_x^2)} + \mu_4,$$

for any subinterval I' of I and any $t' \in I'$. Denote the left side by $X(I')$, Hölder's inequality and Sobolev embedding theorem reveal that

$$\Big\|W|v|^3 + b(x)Wv\Big\|_{L_t^1(I';L_x^2)} \le C\big|I'\big|^{\frac{4}{5}}X(I').$$

Chopping I up to sufficiently small intervals, we have

$$X(I') \leq C(\mu_1);$$

particularly, we conclude

$$\|1_{D>\mu_4^{-2}} v\|_{L_t^5(I;L_x^{10})} \leq C(\mu_1)\mu_4,$$

which yields (3.11) by interpolation inequality thus finishing the proof of Claim 2.

Claim 3. $\|1_{D>\mu_4^{-3}} \int_I S_{2,\alpha}(t-s)(h(u(s))ds\|_{L^2(\mathbb{R}^3)} \lesssim_{\mu_1} \mu_2$.

From finite speed of propagation, $S_{2,\alpha}(t-s)\big(h(u(s))1_{D\leq\mu_4^{-2}}\big)$ is supported in $D \leq \mu_4^{-2} + 4\mu_1^{-1}$. Therefore $1_{D>\mu_4^{-3}} \int_I S_{2,\alpha}(t-s)\big(h(u(s))1_{D\leq\mu_4^{-2}}\big)ds = 0$. Thus it suffices to prove

$$\left\|\int_I S_{2,\alpha}(t-s)\big(h(u(s))1_{D\geq\mu_4^{-2}}\big)ds\right\|_{L^2} \lesssim_{\mu_1} \mu_2. \tag{3.12}$$

By Strichartz estimates, the left is bounded by

$$\||u|^3 1_{D\geq\mu_4^{-2}}\|_{L_t^1 L_x^2} + \|b(x)u 1_{D\geq\mu_4^{-2}}\|_{L_t^1 L_x^2}.$$

Since $D(x) \geq \mu_4^{-2}$ implies $|x| \geq \mu_4^{-2}$, then (3.12) follows from Hölder's inequality, the decay of $b(x)$ and Claim 2.

Step Four. L^2 localization of good times. In this step, we prove the L^2 localization of $u(t_*)$, namely for T_1 sufficiently large, $t > T_1$,

$$\|1_{D>\mu_4^{-3}} u(t_*)\|_2 = O_{L^2}(\mu_1). \tag{3.13}$$

The proof is based on the decay of linear part and Claim 3 in step three. Indeed, from Duhamel principle and Claim 3,

$$1_{D>\mu_4^{-3}} u(t_*) = 1_{D>\mu_4^{-3}} S_{1,\alpha}(\mu_1^{-1})u(t_* - \mu_1^{-1}) + 1_{D>\mu_4^{-3}} S_{2,\alpha}(\mu - 1^{-1})u_t(t_* - \mu_1^{-1}) + O_{L^2}(\mu_2).$$

Then since $S_{1,\alpha}$ and $S_{2,\alpha}$ have a exponential decay, we obtain

$$\begin{aligned}\left\|1_{D>\mu_4^{-3}} u(t_*)\right\|_2^2 &= \left\langle 1_{D>\mu_4^{-3}} u(t_*), S_{1,\alpha}(\mu_1^{-1})u(t_* - \mu_1^{-1})\right\rangle + \left\langle 1_{D>\mu_4^{-3}} u(t_*), S_{2,\alpha}(\mu_1^{-1})\partial_t u(t_* - \mu_1^{-1})\right\rangle + O(\mu_2)\\ &\leq e^{-\mu_1^{-1}\beta}\|u(t_*)\|_2^2 + O(\mu_2) \lesssim \mu_1.\end{aligned}$$

Thus (3.13) follows.

Step Five. L^2 localization of all time. First from Duhamel principle and similar arguments as step four, it is easy to verify,

$$\|1_{D\geq\mu_4^{-3}} u(t)\|_2 \leq \mu_1,$$

for $t\in(t_*,t_*+4\mu_1^{-1})$. Indeed, from Duhamel principle, finite speed of propagation, Claim 2 and Claim 3, we have

$$\begin{aligned}\left\|1_{D>\mu_4^{-4}}u(t)\right\|_2 &\le \left\|1_{D>\mu_4^{-4}}S_{1,\alpha}(t-t_*)u(t_*)\right\|_2+\left\|1_{D>\mu_4^{-4}}S_{2,\alpha}(t-t_*)u(t_*)\right\|_2\\ &\quad+\left\|1_{D>\mu_4^{-4}}\int_{t_*}^{t}e^{-\alpha(t-s)}S_{2,\alpha}(t-s)h(u(s))\right\|_2\\ &\lesssim \left\|1_{D>\mu_4^{-3}}u(t_*)\right\|_2+\left\|1_{D>\mu_4^{-3}}u(t_*)\right\|_2+\mu_2\lesssim\mu_1.\end{aligned}$$

Split the whole interval $[T_1,\infty)$ into subintervals with length $2\mu_1^{-1}$, denote these subintervals as $I_1,I_2,I_3,....$ Denote $t_*\in I_L$ by t_*^L. It is obvious that I_{L+1} is covered by $(t_*^L,t_*^L+4\mu_1^{-1})$, for $L=1,2,....$ Now let's define $\tilde{x}_j(t)$ for each t by the following rule: For $t\in I_{L+1}$, $j\in\{1,...,J\}$, take $\tilde{x}_j(t)=x_j(t_*^L)$. It is direct to see $\tilde{x}_j(t)$ defined above satisfies Proposition 3.4 for all $t>T_1+2\mu_1^{-1}$. □

Proposition 3.5. *Let u be a global solution to (1.1) with $\mathcal{H}$ norm at most $E>0$. Then for $\mu_0>0$, there exit $J=J(E,\mu_0)$ and functions $\tilde{x}_1(t),...,\tilde{x}_J(t):\mathbb{R}^+\to\mathbb{R}^3$, and $\eta=\eta(E,\mu_0)>0$ such that*

$$\limsup_{t\to\infty}\int_{dist(x,\{0,\tilde{x}_1(t),...,\tilde{x}_J(t)\})>\eta^{-1}}|\nabla u|^2+|u|^2+|\partial_t u|^2\le\mu_0.$$

Proof. Choose $\mu_4\ll\mu_3\ll\mu_2\ll\mu_1\ll\mu_0$ as Proposition 3.4. Suppose that $t\in I_{j+1}$, then $0<t-t_*^j<4\mu_1^{-1}$. Define $D(t)=dist(x,\{0,\tilde{x}_1(t),...,\tilde{x}_J(t)\})$. Then the proof of Proposition 3.4 implies that for all $t\in I_{j+1}$,

$$1_{D(t)}=1_{D(t_*^j)}. \tag{3.14}$$

Let χ_1 be a cutoff function supported in $D(t_*^j)>\mu_4^{-4}$, which equals one in $D(t_*^j)>\mu_4^{-5}$, with bound $|\nabla\chi_1|\lesssim\mu_4$. Therefore we have

$$\int_{D(t_*^j)>\mu_4^{-5}}|\nabla u(t)|^2dx\le\|u(t)\chi_1\|_{H^1}+\mu_4.$$

Duhamel principle and finite speed of propagation give

$$\begin{aligned}u(t)\chi_1&=S_{1,\alpha}(\mu_1^{-1})1_{D(t_*^j)>\mu_4^{-3}}u(t-\mu_1^{-1})+S_{2,\alpha}(\mu_1^{-1})1_{D(t_*^j)>\mu_4^{-3}}\partial_t u(t-\mu_1^{-1})\\&\quad+\int_{t-\mu_1^{-1}}^{t}S_{2,\alpha}(t-s)\big(h(u)1_{D(t_*^j)>\mu_4^{-3}}\big)(s)ds.\end{aligned}$$

By Strichartz estimates and exponential decay of $S_{1,\alpha},S_{2,\alpha}$, we get

$$\|u(t)\chi_1\|_{H^1}\le C(E)e^{-\beta\mu_1^{-1}}+\left\|h(u)1_{D(t_*^j)>\mu_4^{-3}}\right\|_{L_t^1((t-\mu_1^{-1},t);L_x^2)}.$$

Then Claim 2, $(t-\mu_1^{-1},t)\subset(t_*^j-4\mu_1^{-1},t_*^j+4\mu_1^{-1})$, and (3.14) imply

$$\int_{D(t)>\mu_4^{-5}}|\nabla u(t)|^2dx \leq \mu_1, \tag{3.15}$$

which gives us the desired bound for $\nabla u(t)$.
Next, we prove the desired bound for $\partial_t u$. By Duhamel principle and finite speed of propagation, we obtain

$$\begin{aligned}1_{D(t_*^j)>\mu_4^{-4}}u(t) =& S_{1,\alpha}(t-t_*^j)1_{D(t_*^j)>\mu_4^{-3}}u(t_*^j)+S_{2,\alpha}(t-t_*^j)1_{D(t_*^j)>\mu_4^{-3}}\partial_t u(t_*^j)\\ &+\int_{t_*^j}^t e^{-\alpha(t-s)}S_{2,\alpha}(t-s)1_{D(t_*^j)>\mu_4^{-3}}h(u(s))ds.\end{aligned}$$

Direct calculations yield

$$\begin{aligned}\left\|\partial_t\left[1_{D(t_*^j)>\mu_4^{-4}}u(t)\right]\right\|_2 \leq& \left\|1_{D(t_*^j)>\mu_4^{-3}}u(t_*^j)\right\|_{H^1}+\left\|1_{D(t_*^j)>\mu_4^{-3}}\partial_t u(t_*^j)\right\|_2\\ &+\left\|1_{D(t_*^j)>\mu_4^{-3}}h(u(s))\right\|_{L_t^1(I;L_x^2)},\end{aligned}$$

where $I=[t_*^j-\mu_1^{-1},t_*^j+\mu_1^{-1}]$. Then Claim 2, (3.7) and (3.15) imply

$$\left\|\partial_t\left[1_{D(t_*^j)>\mu_4^{-4}}u(t)\right]\right\|_2 \lesssim \mu_1.$$

Then the desire bound follows from (3.14). □

From the proof of Proposition 10.1 in T. Tao [37], Proposition 3.3 is a corollary of Proposition 3.5 and the following lemma.

Lemma 3.6. *Let u be a global solution with $\mathcal{H}$ norm at most E. Suppose that we have the energy concentration bound*

$$\int_{|x-x_0|<R}|u(t_0,x)|^2+|\nabla u(t_0,x)|^2+|\partial_t u(t_0,x)|^2dx \geq \eta_1^2$$

for some $x_0\in\mathbb{R}^3$, $t_0\in\mathbb{R}^+$, $R>0$, and sufficiently small $\eta_1>0$. Then, if t_0 is sufficiently large depending on u,E,x_0,R,η_1, we have the improved energy concentration

$$\int_{|x-x_0|<R'}|u(t_0,x)|^2+|\nabla u(t_0,x)|^2+|\partial_t u(t_0,x)|^2dx \geq \beta,$$

for some $\beta>0$ independent of η_1 and some R' depending on E,R,η_1.

The proof of Lemma 3.6 can be reduced to the following lemma.

Lemma 3.7. *Given $E > 0$, let $\eta_1 > 0$ be sufficiently small, there exists $\beta > 0$ with the following property: Suppose that we have the energy concentration bound*

$$\eta_1^2 \leq \int_{|x-x_0|<R} |\nabla u(t_0,x)|^2 + |u(t_0,x)|^2 + |\partial_t u(t_0,x)|^2 \leq \beta$$

for some $x_0 \in \mathbb{R}^3$, $t_0 \in \mathbb{R}^+$, $R > 0$, and some global solution u with $\mathcal{H}$ norm at most E. Then, if t_0 is sufficiently large depending on E, x_0, R, η_1, we have

$$\int_{|x-x_0|<R'} |\nabla u(t_0,x)|^2+|u(t_0,x)|^2+|\partial_t u(t_0,x)|^2 \geq \int_{|x-x_0|<R} |\nabla u(t_0,x)|^2+|u(t_0,x)|^2+|\partial_t u(t_0,x)|^2+\eta_4^2,$$

for some $\eta_4(E,\eta_1) > 0$ and $R'(E,R,\eta_1,\eta_4)$.

Proof. For simplicity, define $e(t,x) = |\nabla u(t,x)|^2 + |u(t,x)|^2 + |\partial_t u(t,x)|^2$. Fix $E > 0$, let $\beta > 0$ be a sufficiently small quantity to be determined. Choose parameters $\eta_1 \gg \eta_2 \gg \eta_3 \gg \eta_4 > 0$. Let $R_0 > max(32R, \frac{32}{\eta_3})$. Suppose our claim fails, then there exists $R' > R_0$ such that

$$\int_{R<|x-x_0|<R'} e(t_0,x)dx \lesssim \eta_4^2,$$

especially, we have

$$\int_{|x-x_0|<R'} e(t_0,x) \lesssim \beta.$$

Choose $\beta < \epsilon$, where ϵ is the constant in Proposition 2.4. Denote the solution of (1.1) with initial data $1_{|x-x_0|<R'} u(t_0,x)$ at t_0 by $\tilde{u}$. Proposition 2.4 implies

$$\|\tilde{u}\|_{\mathcal{H}} \leq e^{-\gamma(t-t_0)} \|\tilde{u}(t_0)\|_{\mathcal{H}} \lesssim \beta. \tag{3.16}$$

Finite speed of propagation implies

$$u(x,t) = \tilde{u}(x,t) \text{ in } \{(x,t) : |x-x_0| < R' - |t-t_0|\}.$$

Consider a time interval $I = [t_0, t_0 + \eta_3^{-1}]$. Then for $t \in I$, we have

$$\int_{|x-x_0|<R'/2} e(t,x)dx = \int_{|x-x_0|<R'/2} \tilde{e}(t,x)dx.$$

Combining with (3.16), we have verified

$$\int_{|x-x_0|<R'/2} e(t+\eta_3^{-1},x)dx \lesssim e^{-\eta_3^{-1}} \beta \lesssim \eta_3.$$

If we have obtained

$$\inf_{t\in I}\int_{|x-x_0|<R'/2} e(t,x) \geq \eta_1^2, \tag{3.17}$$

then contradiction follows. Hence, it suffices to prove (3.17). Let $\psi(x)$ be a smooth cutoff function which equals 1 in $\{|x-x_0| < R'/4\}$, vanishes when $|x-x_0| > R'/2$, with bound $|\nabla\psi(x)| = O(R'^{-1})$, then we have

$$\begin{aligned}
&\frac{d}{dt}\int_{R^3}\psi(x)\left[|\nabla u|^2+|u|^2+|u_t|^2\right]dx \\
&= 2\int_{R^3}\psi(x)u_t h(u)dx - 4\alpha\int_{R^3}\psi(x)|u_t|^2dx - 2\int_{R^3}\nabla\psi(x)\nabla u u_t dx.
\end{aligned}$$

Therefore, Hölder's inequality yields,

$$\begin{aligned}
&\left|\int_{\mathbb{R}^3}\psi(x)e_\mu(t)dx - \int_{\mathbb{R}^3}\psi(x)e(t_0)dx\right| \\
&\leq \int_{t_0}^{t_0+1/\eta_3}\left|\frac{d}{dt}\int_{\mathbb{R}^3}\psi(x)edx\right|dt \\
&\leq C\int_{t_0}^{t_0+1/\eta_3}\|u_t\|_2\|h(u)\|_2 + \|u_t\|_2^2 + \|u_t\|_2\|\nabla u\|_2 dt.
\end{aligned}$$

Sobolev embedding implies $\|h(u)\|_2 \leq C(E)$, thus

$$\left|\int_{\mathbb{R}^3}\psi(x)e(t)dx - \int_{\mathbb{R}^3}\psi(x)e(t_0)dx\right| \leq C(E)\int_{t_0}^{t_0+1/\eta_3}\Big(\|u_t\|_2^2+\|u_t\|_2\Big)dt. \tag{3.18}$$

Since $\int_0^\infty \|u_t\|_2^2\,dt < \infty$, choose t_0 sufficiently large such that

$$\int_{t_0}^{t_0+1/\eta_3}\|u_t\|_2^2dt \leq \sqrt{\eta_3}\eta_1^6,$$

then

$$\int_{|x|<R'/2} e(t)dx \geq \int_{\mathbb{R}^3}\psi(x)e(t)dx \geq \int_{\mathbb{R}^3}\psi(x)e(t_0)dx - \sqrt{\eta_3}\eta_1^3 \gtrsim \eta_1^2,$$

thus proving (3.17), from which our lemma follows. □

4 Existence of Attractors

4.1 Global attractor

In the defocusing case, the existence of the global attractor is known, for instance [30], we summarize the result for our convenience.

Lemma 4.1. *If $\lambda > 0$, for $b(x)$ in Theorem 1.1, for any initial data $(u_0, u_1) \in \mathcal{H}$, there exists some $t_0 > 0$ such that the trajectory $\underset{t \geq t_0}{\cup} (u(\cdot, t), \partial_t u(\cdot, t))$ is pre-compact in $\mathcal{H}$, and the ω-limit set of $u(t)$ consists of equilibriums to (1.1).*

Remark 4.1. *In the focusing case, if we consider radial data, Lemma 3.6 is not needed, hence for $b(x)$ in Theorem 1.1, by Proposition 8.1 in T. Tao [37] and Proposition 3.3 above, we can prove for any $\mu > 0$, there exists $R(\mu)$ such that*

$$\limsup_{t \to \infty} \int_{|x| > R(\mu)} |\nabla u(t, x)|^2 + |u(t, x)|^2 + |\partial_t u(t, x)|^2 dx \leq \mu.$$

Thus by Proposition 4.2 below, we can also get the pre-compactness of the trajectory.

4.2 Concentration compact attractor

In this subsection, we consider the focusing case. Recall the definition of concentration-compact attractor as follows:

Definition 4.1. *Given any $h \in \mathbb{R}^3$, let $\tau_h : \mathcal{H} \to \mathcal{H}$ be the shift operator $\tau_h(f(x), g(x)) = (f(x-h), g(x-h))$, and we denote the translation group by $G = \{\tau_h : h \in \mathbb{R}^3\}$. Given any $K \subseteq \mathcal{H}$, we denote the orbit of K by $GK = \{gf : g \in G, f \in K\}$. If $GK = K$, then we call K G-invariant. Suppose that $J \geq 0$ is an integer, we let*

$$JK \equiv \{f_1 + ... + f_J : f_1, f_2, ..., f_J \in K\}.$$

We say $E \subseteq \mathcal{H}$ is G-precompact with J components if $E \subseteq J(GK)$ for some compact $K \subseteq \mathcal{H}$ and $J \geq 1$.

We recall the following criterion for compact attractors proved by Proposition B.2 in Tao [37].

Proposition 4.2. *Let $\mathcal{U}$ be a collection of trajectories $u : \mathbb{R}^+ \to \mathcal{H}$. If $\mathcal{U}$ is bounded in $\mathcal{H}$, and for any $\mu_0 > 0$ there exists $\mu_1 > 0$ such that*

$$\limsup_{t \to \infty} \left\| P_{>1/\mu_1} u(t) \right\|_{\mathcal{H}} \leq \mu_0,$$
$$\limsup_{t \to \infty} \int_{|x| > 1/\mu_1} |u(x, t)|^2 + |\nabla u(x, t)|^2 + |\partial_t u(x, t)|^2 dx \leq \mu_0.$$

Then there exists a compact set $K \subset \mathcal{H}$ such that $\lim_{t \to \infty} dist_{\mathcal{H}}(u(t), K) = 0$.

Proposition 4.3. *Let $\mathcal{U}$ be a collection of trajectories $u : \mathbb{R}^+ \to \mathcal{H}$, and let $J \geq 1$. If $\mathcal{U}$ is bounded in $\mathcal{H}$, and for any $\mu_0 > 0$ there exists $\mu_1 > 0$ such that for every $u \in \mathcal{U}$ we have*

$x_1, ..., x_J : \mathbb{R}^+ \to \mathbb{R}^3$ for which

$$\limsup_{t\to\infty} \left\|P_{>1/\mu_1} u(t)\right\|_{\mathcal{H}} \leq \mu_0,$$
$$\limsup_{t\to\infty} \int_{dist(x,\{x_1(t),x_2(t),...,x_J(t)\})>1/\mu_1} |u(x,t)|^2 + |\nabla u(x,t)|^2 + |\partial_t u(x,t)|^2 dx \leq \mu_0.$$

Then there exists a G-precompact set $K \subset \mathcal{H}$ with J components such that $\lim_{t\to\infty} dist_{\mathcal{H}}(u(t), K) = 0$.

Proof. Although the proof is almost the same as proposition B.3 of T. Tao [37], for reader's convenience, we give a sketch here. We use the partition of unity

$$1 = \sum_{j=1}^{J} \psi_{j,t}(x),$$

where

$$\psi_{j,t}(x) \equiv \frac{\langle x - x_j(t)\rangle^{-1}}{\sum\limits_{l=1}^{J} \langle x - x_l(t)\rangle^{-1}}.$$

Split $(u(x,t), \partial_t u(x,t))$ as

$$u(t) = \sum_{j=1}^{J} \tau_{x_j(t)} w_j(t), \partial_t u(t) = \sum_{j=1}^{J} \tau_{x_j(t)} v_j(t), \tag{4.1}$$

where

$$w_j(t) = \tau_{-x_j(t)}\psi_{j,t}(x)u(t),\ \ v_j(t) = \tau_{-x_j(t)}\psi_{j,t}(x)\partial_t u(t).$$

The localization of u and $\partial_t u$ implies for any $\mu_0 > 0$, there exists $\eta > 0$ such that

$$\limsup_{t\to\infty} \|P_{>\eta} w_j\|_{H^1} + \|P_{>\eta} v_j\|_{L^2} \leq \mu_0, \quad \limsup_{t\to\infty} \int_{|x|>\eta} |w_j|^2 + |\nabla w_j|^2 + |v_j|^2 \leq \mu_0.$$

From Proposition 4.2, there exist a compact set $K_1 \subset H^1$ and a compact set $K_2 \subset L^2$, such that

$$\lim_{t\to\infty} dist(w_j(t), K_1) = 0, \quad \lim_{t\to\infty} dist(v_j(t), K_2) = 0,$$

for all $j = 1, 2, ..., J$. Combining with (4.1), we obtain

$$dist_{\mathcal{H}}(u, J(GK)) = 0,$$

where $K = K_1 \times K_2$. □

As a corollary of Proposition 4.3, Proposition 3.3, Lemma 3.2, we have

Corollary 4.4. *There exists a compact set $K \subset \mathcal{H}$ and $0 \le J < \infty$, such that*

$$\lim_{t\to\infty} dist_{\mathcal{H}}(u(t), J(GK)) = 0.$$

5 Proof of theorem 1.1

In this section we aim to finish the proof of Theorem 1.1. As a preparation, we give some lemmas.

Lemma 5.1. *Define $L = -\Delta + I + V(x)$, where $V(x)$ satisfies $|\partial_x^k V(x)| \le C\langle x\rangle^{-\mu}$, for $|k| \le 1$ and some $\mu > 3$. Then $R(L)$ is a closed subspace of L^2.*

Proof. Take H^2 as the domain of L, then L is self-adjoint in L^2 and by Birman-Schwinger criterion the kernel of L is finitely dimensional. We first show it suffices to prove for any $f \in N(L)^\perp \cap H^2$,

$$\|(-\Delta + I)f\|_2 \lesssim \|Lf\|_2. \tag{5.1}$$

Indeed, for any fixed $g \in \overline{R(L)}$, there exists $g_n \in H^2$ such that

$$\lim_{n\to\infty} Lg_n = g. \tag{5.2}$$

Without loss of generality, we assume $g_n \in N(L)^\perp$. According to (5.1),

$$\|(-\Delta + I)g_n\|_2 \lesssim \|Lg_n\|_2 \lesssim \|g\|_2.$$

Thus $\{g_n\}$ is bounded in H^2, then up to a subsequence, we can assume $g_n \rightharpoonup g_*$ weakly in H^2, as $n \to \infty$. By (5.2), we conclude $Lg_* = g$. Hence $R(L)$ is closed in L^2.

Now, we prove (5.1). We verify it by a contradiction argument. Suppose that (5.1) is false, then there exists $\{f_n\} \subset H^2 \cap N(L)^\perp$ such that

$$\|(-\Delta + I)f_n\|_2 \ge n\,\|Lf_n\|_2\,.$$

Without loss of generality, we assume $\|(-\Delta + I)f_n\|_2 = 1$. Then $\lim\limits_{n\to\infty} \|Lf_n\|_2 = 0$, which is equivalent to

$$\lim_{n\to\infty} \|(-\Delta + I)f_n\|_2^2 - \langle(-\Delta + I)f_n, Vf_n\rangle - \langle Vf_n, (-\Delta + I)f_n\rangle + \|Vf_n\|_2^2 = 0. \tag{5.3}$$

Since $\|f_n\|_{H^2}$ is bounded, after extracting a subsequence, we may assume $f_n \rightharpoonup f_*$ weakly in H^2. Since $N(L)^\perp$ is closed respect to weak limit, we have $f_* \in N(L)^\perp$. We claim

$$\lim_{n\to\infty} \langle f_n, Vf_n\rangle = \langle f_*, Vf_*\rangle\,, \quad \lim_{n\to\infty} \langle \Delta f_n, Vf_n\rangle = \langle \Delta f_*, Vf_*\rangle\,, \quad \lim_{n\to\infty} \|Vf_n\|_2^2 = \|Vf_*\|_2^2\,. \tag{5.4}$$

By integrating by parts,

$$\int_{\mathbb{R}^3} \Delta f_n V f_n \,\mathrm{d}x = -\int_{\mathbb{R}^3} \nabla f_n \cdot \nabla V f_n \,\mathrm{d}x - \int_{\mathbb{R}^3} V \nabla f_n \cdot \nabla f_n \,\mathrm{d}x.$$

For any $\varepsilon > 0$, choosing $R > 0$ sufficiently large, Hölder's inequality and Sobolev embedding give

$$\left| \int_{|x|\geq R} \nabla f_n \nabla V f_n \, dx \right| \leq \frac{1}{R} \int_{|x|\geq R} |\nabla f_n| \, |x \nabla V| \, |f_n| \,\mathrm{d}x \leq \frac{1}{R} \|\nabla f_n\|_2 \|f_n\|_2 \| |x| \, \nabla V\|_\infty \lesssim \frac{1}{R} < \varepsilon. \tag{5.5}$$

Similarly we have

$$\left| \int_{|x|\geq R} V^2 |f_n|^2 \,\mathrm{d}x \right| \leq \frac{1}{R} \|f_n\|_2^2 \left\| |x| \, V^2 \right\|_\infty \lesssim \frac{1}{R} < \varepsilon, \tag{5.6}$$

$$\left| \int_{|x|\geq R} V |\nabla f_n|^2 \,\mathrm{d}x \right| \leq \frac{1}{R} \|\nabla f_n\|_2^2 \, \| |x| \, V\|_\infty \lesssim \frac{1}{R} < \varepsilon. \tag{5.7}$$

Since the Sobolev embedding is compact on bounded domains, by extracting a subsequence, together with (5.5), (5.6) and (5.7), we obtain

$$\int V {f_n}^2 \,\mathrm{d}x \to \int V f_*^2 \,\mathrm{d}x, \quad \int \nabla f_n \cdot \nabla V f_n \,\mathrm{d}x \to \int \nabla f_* \cdot \nabla V f_* \,\mathrm{d}x,$$
$$\int V \nabla f_n \cdot \nabla f_n \,\mathrm{d}x \to \int V \nabla f_* \cdot \nabla f_* \,\mathrm{d}x, \quad \|V f_n\|_{L^2}^2 \to \|V f_*\|_{L^2}^2, \qquad \text{as } n \to \infty.$$

Then (5.4) follows. Therefore, we have proved

$$\begin{aligned} \lim_{n\to\infty} - \langle(-\Delta + I)f_n, V f_n\rangle - \langle V f_n, (-\Delta + I)f_n\rangle + \|V f_n\|_2^2 = &- \langle(-\Delta + I)f_*, V f_*\rangle - \langle V f_*, (-\Delta + I)f_*\rangle \\ &+ \|V f_*\|_2^2 . \end{aligned} \tag{5.8}$$

Combining (5.3) and (5.8), with $\liminf_{n\to\infty} \|(-\Delta + I)f_n\|_2^2 \geq \|(-\Delta + I)f_*\|_2^2$, we have

$$\|(-\Delta + I)f_*\|_2^2 - \langle(-\Delta + I)f_*, V f_*\rangle - \langle V f_*, (-\Delta + I)f_*\rangle + \|V f_*\|_2^2 \leq 0.$$

Hence $L f_* = 0$. By $f_* \in N(L)^\perp$, we have $f_* = 0$. Then (5.3) and (5.8) give

$$\lim_{n\to\infty} \|(-\Delta + I)f_n\|_2 = 0,$$

which contradicts with $\|(-\Delta + I)f_n\|_2 = 1$. Therefore we have verified (5.1). $\square$

Lemma 5.2. *For $f \in \big(P_c(L)L^2\big) \cap H^2$, we have*

$$\|f\|_{H^1}^2 \lesssim |\langle Lf, f\rangle| \,. \tag{5.9}$$

Proof. We prove (5.9) by contradiction. If it fails, as the proof of (5.1), there exits $\{f_n\} \subset H^1 \cap P_c(L)L^2$, such that $\|f_n\|_{H^1} = 1$, $f_n \rightharpoonup f_*$ weakly in H^1. Moreover,

$$\lim_{n\to\infty} \left|\|f_n\|_{H^1}^2 + \langle Vf_*, f_*\rangle\right| = 0 \tag{5.10}$$

$$|\langle Lf_*, f_*\rangle| = 0. \tag{5.11}$$

Since $P_c(L)L^2$ is closed under weak limit, $f_* \in P_c(L)L^2$. $\sigma_c(L) = (1, \infty)$, (5.11) and spectrum decomposition theorem imply

$$|\langle Lf_*, f_*\rangle| \geq \|f_*\|_2^2,$$

Hence, $f_* = 0$, which combined with (5.10) yields $\lim\limits_{n\to\infty} \|f_n\|_{H^1}^2 = 0$, contradicting with $\|f_n\|_{H^1} = 1$. □

We recall the Lojasiewicz inequality for the analytic function defined on $\mathbb{R}^d$.

Lemma 5.3. *Suppose that $F : \mathbb{R}^d \to \mathbb{R}$ is an analytic function near its critical point a (i.e. $\nabla F(a) = 0$). Then there exists a positive constant σ and $\theta \in (0, \frac{1}{2})$ depending on a such that when $|x - a| \leq \sigma$,*

$$|F(x) - F(a)|^{1-\theta} \leq |\nabla F(x)|.$$

The following type inequality, which is now called the Lojasiewicz-Simon inequality, is an extension of Lojasiewicz inequality to the infinite-dimensional space.

Lemma 5.4. *Let $\mathcal{E}$ be the set of equilibrium points of (1.1):*

$$\mathcal{E} = \{Q : -\Delta Q + Q + b(x)Q + \lambda Q^3 = 0\}$$

and $Q \in \mathcal{E}$. Then there exit a small positive constant σ and $\theta \in (0, \frac{1}{2})$ depending on Q such that for all $u \in H^2$, $\|u - Q\|_{H^1} \leq \sigma$, we have

$$\| - \Delta u + u + b(x)u + \lambda u^3\|_{H^{-1}} \geq |J(u) - J(Q)|^{1-\theta},$$

where $J(u)$ is the stationary energy.

Proof. Consider the stationary equation:

$$-\Delta Q + Q + b(x)Q + \lambda Q^3 = 0. \tag{5.12}$$

The linearized equation of (5.12) near the equilibrium Q is given by

$$Lw = -\Delta w + w + b(x)w + 3\lambda Q^2 w.$$

Define $V = b(x) + \lambda 3Q^2$, by Lemma 2.6, Q is of exponential decay, thus V satisfies Lemma 5.1. Therefore we have the decomposition $L^2 = N(L) \bigoplus \overline{R(L)} = N(L) \bigoplus R(L)$, which implies that

$$Lw = h, \quad w \in H^2,$$

is solvable if and only if $h \in N(L)^{\perp}$. Let $(\varphi_1, ..., \varphi_m)$ be the normalized orthogonal basis of $N(L)$ in L^2, and Π be the projection from L^2 onto $N(L)$. Define $\mathcal{L} : H^2 \to L^2$ by

$$\mathcal{L}w = \Pi w + Lw.$$

Then $\mathcal{L}$ is a one-to-one and onto operator. Moreover, $\mathcal{L}$ can be viewed as a bijection operator from H^1 to H^{-1}. In fact, if $\mathcal{L}v = 0$, for some $v \in H^1$, then $-\Delta v + v = -V(x)v - \Pi v \in L^2$. Thus $v \in H^2$, by the injection of $\mathcal{L} : H^2 \to L^2$, we have $v = 0$. It remains to prove $\mathcal{L} : H^1 \to H^{-1}$ is surjective. For any $g \in H^{-1}$, there exists $g_n \in L^2$ such that $g_n \to g$ in H^{-1}. For each g_n, since $\mathcal{L}$ is surjective from H^2 to L^2, there exists $v_n \in H^2$ such that $\mathcal{L}v_n = \Pi v_n + Lv_n = g_n$. Decompose v_n into

$$v_n = \sum_{j=1}^{N} \Pi_j v_n + \Pi v_n + \zeta_n,$$

where $\zeta_n \in P_C(L)L^2$, Π_j is the projection to the eigenfunction space of the j-th nonzero eigenvalue λ_j of L. This it possible by Birman-Schwinger criterion and Agmon-Kuroda theory. From $\sum_{j=1}^{N} \lambda_j \Pi_j v_n + L\zeta_n + \Pi v_n = g_n$, we get

$$\begin{aligned} \langle g_n, \Pi v_n \rangle &= \langle \Pi v_n, \Pi v_n \rangle \\ \langle g_n, \Pi_j v_n \rangle &= \lambda_j \langle \Pi_j v_n, \Pi_j v_n \rangle \\ \langle L\zeta_n, \zeta_n \rangle &= \langle \zeta_n, g_n \rangle, \end{aligned} \tag{5.13}$$

which combined with (5.9), $\|\Pi_j f\|_{L^2} \sim \|\Pi_j f\|_{H^1}$, $\|\Pi f\|_{L^2} \sim \|\Pi f\|_{H^1}$ (notice the eigenfunction space is finitely dimensional) yields

$$\|g_n\|_{H^{-1}} \gtrsim \sum_{j=1}^{N} \|\Pi_j v_n\|_{H^1} + \|\Pi v_n\|_{H^1} + \|\zeta_n\|_{H^1} \geq \|v_n\|_{H^1}.$$

Therefore, we deduce $\{v_n\}$ is a Cauchy sequence in H^1. Thus for some $v \in H^1$, $v_n \to v$ in H^1, as a consequent of $\mathcal{L}v_n \to g$ in H^{-1}, we obtain $\mathcal{L}v = g$ in distribution, which yields $\mathcal{L} : H^1 \to H^{-1}$

is a bijection. Let $u = v + Q$, define $M : H^1 \to H^{-1}$ by

$$M(v) = -\Delta u + u + b(x)u + \lambda u^3 : H^1 \to H^{-1}.$$

Then the Frechet derivative of M at $v = 0$ is $DM(0) = L$. Let $\mathcal{N}v = M(v) + \Pi v$, we have $D\mathcal{N}(0) = \mathcal{L}$. Since $\mathcal{L}$ is a one-to-one and onto operator from H^1 to H^{-1}, by implicit theorem, there is a neighborhood $U_1(0)$ of the origin in H^1 and a neighborhood $U_2(0)$ of the origin in H^{-1}, and an inverse mapping

$$\Psi : U_2(0) \mapsto U_1(0),$$

such that

$$\mathcal{N}\big(\Psi(g)\big) = g, \ \forall g \in U_2(0), \tag{5.14}$$

and

$$\Psi\big(\mathcal{N}(v)\big) = v, \ \forall v \in U_1(0). \tag{5.15}$$

It is easy to see the operators M, $\mathcal{N}$, and Ψ are analytic. Furthermore, there exists $C > 0$ such that

$$\|\Psi(g_1) - \Psi(g_2)\|_{H^1} \le C\|g_1 - g_2\|_{H^{-1}}, \ \forall g_1, g_2 \in U_2(0), \tag{5.16}$$

and

$$\|\mathcal{N}(v_1) - \mathcal{N}(v_2)\|_{H^{-1}} \le C\|v_1 - v_2\|_{H^1}, \ \forall v_1, v_2 \in U_1(0), \tag{5.17}$$

which combined with $M(v) = \mathcal{N}(v) - \Pi v$ gives

$$\|M(v_1) - M(v_2)\|_{H^{-1}} \le C\|v_1 - v_2\|_{H^1}. \tag{5.18}$$

Let $\xi = (\xi_1, ..., \xi_m) \in \mathbb{R}^m$. For ξ sufficiently small, $\sum\limits_{j=1}^{m} \xi_j\varphi_j \in U_2(0)$. Define $\Gamma : \mathbb{R}^m \to \mathbb{R}$ as follows:

$$\Gamma(\xi) = J(\Psi(\sum_{j=1}^{m} \xi_j\varphi_j) + Q).$$

Then $\Gamma(\xi)$ is analytic in a small neighborhood of the origin in $\mathbb{R}^m$. (5.14) implies for $g \in U_2(0)$, we have

$$\big(D\mathcal{N}\big)(\Psi(g))\big(D\Psi\big)(g) = I.$$

Direct calculations show

$$\frac{\partial \Gamma}{\partial \xi_i} = (DJ)\,(\Psi(\sum_{j=1}^{m} \xi_j \varphi_j) + Q)\,(D\Psi)\,\varphi_j. \tag{5.19}$$

Meanwhile,

$$(DJ)(w)v = \int (-\Delta w + w + b(x)w + \lambda w^3) v dx.$$

Thus $\xi = 0$ is a critical point of $\Gamma(\xi)$. Then (5.19) yields

$$\left|\frac{\partial \Gamma}{\partial \xi_i}\right| \le \|M(\Psi(\sum_{j=1}^{m} \xi_j \varphi_j))\|_{H^{-1}} \|D\Psi(\sum_{j=1}^{m} \xi_j \varphi_j)\|_{L(H^{-1};H^1)} \|\varphi_j\|_2 \le C\|M(\Psi(\sum_{j=1}^{m} \xi_j \varphi_j))\|_{H^{-1}}, \tag{5.20}$$

where we have used $\sum_{j=1}^{m} \xi_j \varphi_j \in U_2(0)$, Ψ is analytic in $U_2(0)$, $\|\varphi_j\|_2 = 1$. For any fixed $v \in U_1(0)$, then there exists ξ sufficiently small, such that $\Pi v = \sum_{j=1}^{m} \xi_j \varphi_j$. Then (5.15), (5.16), (5.17), (5.20) give

$$\begin{aligned} |\nabla \Gamma(\xi)| &\lesssim \|M(\Psi(\Pi v))\|_{H^{-1}} \le \|M(\Psi(\Pi v)) - M(v)\|_{H^{-1}} + \|M(v)\|_{H^{-1}} \\ &\lesssim \|\Psi(\Pi v) - \Psi(\mathcal{N}(v))\|_{H^1} + \|M(v)\|_{H^{-1}} \\ &\lesssim \|\Pi v - \mathcal{N}(v)\|_{H^{-1}} + \|M(v)\|_{H^{-1}} \\ &\lesssim \|M(v)\|_{H^{-1}}. \end{aligned} \tag{5.21}$$

On the other hand, for $t \in [0, 1]$, when $v \in U_1(0)$, $v + t(\Psi(\Pi v) - v) \in U_1(0)$. For $u = v + Q$, similar arguments as (5.21) imply

$$\begin{aligned} |J(u) - \Gamma(\xi)| &= |J(u) - J(\Psi(\Pi v) + Q)| \\ &= \left| \int_0^1 \frac{d}{dt} J(u + (1-t)(\Psi(\Pi v) - v)) dt \right| \\ &= \left| \int_0^1 DJ \cdot (\Psi(\Pi v) - v)) dt \right| \\ &\lesssim \max_{0 \le t \le 1} \|M(v + (1-t)(\Psi(\Pi v) - v))\|_{H^{-1}} \|\Psi(\Pi v) - v\|_{H^{-1}} \\ &\lesssim (\|M(v)\|_{H^{-1}} + \|\Psi(\Pi v) - v\|_{H^1}) \|\Psi(\Pi v) - v\|_{H^{-1}} \\ &\lesssim \|M(v)\|_{H^{-1}}^2. \end{aligned} \tag{5.22}$$

Lojasiewicz inequality, namely Lemma 5.3 implies for ξ sufficiently small and $\theta \in (0, \frac{1}{2})$ such that

$$|\nabla \Gamma(\xi)| \ge |\Gamma(\xi) - \Gamma(0)|^{1-\theta},$$

which means

$$|\nabla\Gamma(\xi)| \geq |\Gamma(\xi) - J(Q)|^{1-\theta}. \tag{5.23}$$

Applying Young inequality, (5.23), (5.21) and (5.22) yield our lemma. □

5.1 The end of the proof

Step one. Combining Corollary 4.4 with Lemma B.7 in Tao [37], we have for any $t_n \to \infty$, up to a subsequence there exits $J_1, J_2, ..., J_M$ and $w_m \in J_m(GK)$ such that

$$u(t_n) = \sum_{m=1}^{M} \tau_{x_{m,n}} w_m + o_{H^1}(1) \tag{5.24}$$

$$\partial_t u(t_n) = \sum_{m=1}^{M} \tau_{x_{m,n}} v_m + o_{L^2}(1), \tag{5.25}$$

where $x_{m,n} \in \mathbb{R}^3$ and they satisfies the separation property: $\lim_{n\to\infty} |x_{m,n} - x_{k,n}| = \infty$, for $k \neq m$.
Step two. By linear energy decoupling property, we have $\sup_m \|(w_m, v_m)\|_{\mathcal{H}} < C$, by the local theory, there exists $T > 0$ such that the solution $W_{m,n}$ to (1.1) with initial data $(w_m(x - x_{m,n}), v_m(x - x_{m,n}))$ is wellposed on $[0, T]$. From the perturbation theorem and the separation of $x_{m,n}$, we obtain

$$\partial_t u(t_n + t) = \sum_{j=1}^{M} \partial_t W_{m,n}(x, t) + o_{L^\infty([0,T];L^2)}(1). \tag{5.26}$$

Indeed in order to prove (5.26), we can first assume (w_m, v_m) are compactly supported, and apply the perturbation theorem, then the general case follows from a density argument and the local well-posedness lemma. Similar techniques give the further decoupling property

$$\int_{t_n}^{T+t_n} \|\partial_s u(s)\|_{L_x^2}^2 ds = \sum_{m=1}^{M} \int_0^T \|\partial_t W_{m,n}(x,t)\|_{L_x^2}^2 dt + o_n(1). \tag{5.27}$$

In order to make (5.27) work, we need to distinguish the different asymptotic behaviors of $W_{m,n}(x, t)$ as $n \to \infty$. This is presented in the following lemma.

Lemma 5.5. *Suppose that $T > 0$ is sufficiently small. If $\lim_{n\to\infty} x_{m,n} = a \in \mathbb{R}^3$, then as $n \to \infty$, we have*

$$W_{m,n}(x,t) \to W_m(x,t), \quad \text{in} \quad L^\infty([0,T]; \mathcal{H}), \tag{5.28}$$

where $W_m(x,t)$ is a solution to (1.1) with data $(w_m(x-a), v_m(x-a))$. If $\lim_{n\to\infty} |x_{m,n}| = \infty$, then

as $n \to \infty$, we have

$$W_{m,n}(x + x_{m,n}, t) \to W_m(x,t), \quad \text{in} \quad L^\infty([0,T];\mathcal{H}), \tag{5.29}$$

where W_m is a solution to

$$\begin{cases} \partial_t^2 W_m - \Delta W_m + W_m + 2\alpha\partial_t W_m + \lambda|W_m|^2 W_m = 0 \\ W_m(x,0) = w_m(x), \partial_t W_m(x,0) = v_m(x) \end{cases}$$

Proof. (5.28) follows directly from the local well-posedness lemma. It suffices to prove (5.29). It is easy to see $U_{m,n}(x,t) = W_{m,n}(x + x_{m,n})$ satisfies

$$\begin{cases} \partial_t^2 U_{m,n} - \Delta U_{m,n} + U_{m,n} + 2\alpha\partial_t U_{m,n} + b(x + x_{m,n})U_{m,n} + \lambda|U_{m,n}|^2 U_{m,n} = 0 \\ U_{m,n}(x,0) = w_m(x), \partial_t U_{m,n}(x,0) = v_m(x) \end{cases} \tag{5.30}$$

We first prove as $n \to \infty$

$$b(x + x_{m,n})U_{m,n} \to 0 \quad \text{in} \quad L^\infty([0,T];L^2). \tag{5.31}$$

Indeed, if (w_m, v_m) is supported in $\{x \in \mathbb{R}^3 : |x| \leq R_0\}$, then $U_{m,n}$ is supported in $\{x \in \mathbb{R}^3 : |x| \leq R_0 + T\}$ for $t \in [0,T]$. Then $\|b(x + x_{m,n})U_{m,n}\|_{L^2_x}$ is bounded by

$$\|b(x + x_{m,n})\|_{L^\infty_{|x|\leq R_0+T}} \|U_{m,n}\|_{L^\infty([0,T];L^2_x)}.$$

By local well-posedness lemma, $\|U_{m,n}\|_{L^\infty([0,T];L^2_x)}$ is bounded by $\|(w_m, v_m)\|_{\mathcal{H}}$, which is uniformly bounded respect to (m,n) by the linear energy decoupling property. Thus from the decay of $b(x)$, for any $\varepsilon > 0$, there exists $N(R_0, \varepsilon)$ sufficiently large such that for $n > N(R_0, \varepsilon)$, $\|b(x + x_{m,n})U_{m,n}\|_{L^\infty([0,T];L^2_x)} < \varepsilon$. For general (w_m, v_m), choosing $(w_{j,m}, v_{j,m})$ which is compactly supported and approximates (w_m, v_m) in $\mathcal{H}$ as $j \to \infty$. Then (5.31) follows from the above discussions and the fact that

$$W_{m,n,j}(x,t) \to W_{m,n}(x,t) \quad \text{in} \quad L^\infty([0,T];\mathcal{H}). \tag{5.32}$$

where $W_{m,n,j}$ is the solution to (1.1) with data $(w_{j,m}(x - x_{m,n}), v_{j,m}(x - x_{m,n}))$. Indeed, (5.32) follows directly from the local well-posedness lemma, and the same holds for $U_{m,n,j}$ defined as $W_{m,n,j}(x + x_{m,n}, t)$.

Second, we prove (5.30). Let U_m be a solution to

$$\begin{cases} \partial_t^2 U_m - \Delta U_m + U_m + 2\alpha\partial_t U_m + \lambda|U_m|^2 U_m = 0 \\ U_m(x,0) = w_m(x), \partial_t U_m(x,0) = v_m(x) \end{cases}$$

Then Duhamel principle shows

$$U_{m,n}(t) - U_m(t) = \int_0^t S_{2,\alpha}(t-s)\lambda\left(|U_m|^2 U_m(s) - |U_{m,n}|^2 U_{m,n}(s)\right) ds + \int_0^t S_{2,\alpha}(t-s) b(x) U_{m,n}(s) ds.$$

Hence we have from Strichartz estimates and Sobolev's inequality that

$$\begin{aligned}
&\|U_{m,n}(t) - U_m(t)\|_{L^\infty([0,T];\mathcal{H})} \\
&\lesssim T\|b(x)U_{m,n}\|_{L^\infty([0,T];L^2_x)} + \int_0^T \|h(U_{m,n}(t)) - h(U_m(t))\|_{L^2_x} dt \\
&\lesssim T\|b(x)U_{m,n}\|_{L^\infty([0,T];L^2_x)} + \int_0^T \|U_{m,n}(t) - U_m(t)\|_{L^6_x}\left(\|U_{m,n}\|^2_{L^6_x} + \|U_{m,n}\|^2_{L^6_x}\right) dt \\
&\lesssim T\|b(x)U_{m,n}\|_{L^\infty([0,T];L^2_x)} + C(E)T\|U_{m,n}(t) - U_m(t)\|_{L^\infty([0,T];H^1_x)}.
\end{aligned}$$

Taking $T > 0$ sufficiently small, we have

$$\|U_{m,n}(t) - U_m(t)\|_{L^\infty([0,T];\mathcal{H})} \lesssim T\|b(x)U_{m,n}\|_{L^\infty([0,T];L^2_x)},$$

therefore (5.30) follows from (5.31). □

Combining (5.27), Lemma 5.5 and $\int_0^\infty \|\partial_t u(t)\|^2_{L^2_x} dt < \infty$, we have

Lemma 5.6. *For the decomposition (5.24) and (5.25), up to a subsequence, for $m \in \{1, ..., M\}$, if $\lim_{n\to\infty} x_{m,n} = a_m \in \mathbb{R}^3$, then we can take $\big(w_m(x - x_{m,n}), v_m(x - x_{m,n})\big) = \big(Q_m, 0\big)$, where Q_m satisfies $-\Delta Q_m + Q_m + b(x)Q_m + \lambda Q_m^3 = 0$; if $\lim_{n\to\infty} |x_{m,n}| = \infty$, then we can take $\big(w_m(x - x_{m,n}), v_m(x - x_{m,n})\big) = \big(Q_m(x - x_{m,n}), 0\big)$, where Q_m satisfies $-\Delta Q_m + Q_m + \lambda Q_m^3 = 0$.*

Therefore, we have proved

Corollary 5.7. *For any sequence $t_n \to \infty$, up to a subsequence, there exist a finite number of equilibrium points Q_m and $x_{m,n} \in \mathbb{R}^3$ such that*

$$u(t_n) = \sum_{m=1}^{M} Q_m(x - x_{m,n}) + o_{H^1}(1). \quad \partial_t u(t_n) = o_{L^2}(1), \tag{5.33}$$

where $(Q_m, x_{m,n})$ satisfies
(i) either $-\Delta Q_m + Q_m + \lambda Q_m^3 = 0$, $\lim_{n\to\infty} |x_{m,n}| = \infty$;
(ii) or $-\Delta Q_m + Q_m + b(x)Q_m + \lambda Q_m^3 = 0$, $x_{m,n} = 0$.
Moreover, $x_{m,n}$ still satisfies the decoupling property.

By contradiction arguments, it is easy to show

$$\lim_{t\to\infty} \partial_t u(t) = 0, \quad \text{in } L^2.$$

Hence the part (I) of Theorem 1.2 is obtained.

Step three. Now we prove part (II) of Theorem 1.2 and Theorem 1.1.

Case 1. If $\lambda > 0$, then by Lemma 4.1, for any sequence $t_n \to \infty$, up to a subsequence, there exists one equilibrium point Q to (1.1) such that

$$u(t_n) = Q(x) + o_{H^1}(1). \quad \partial_t u(t_n) = o_{L^2}(1).$$

This is the pre-compactness of the trajectory, which is important to apply gradient systems theory.

Case 2. We consider $\lambda < 0$. Suppose that there exits non-zero Q_m which is a solution to $-\Delta Q_m + Q_m + \lambda Q_m^3 = 0$. From the energy decoupling property and Lemma 2.6, we have

$$E(u(t_n), 0) \geq E(Q_m, 0), \text{for some } m \in 1, 2, .., M.$$

Since $E(u(t)) \leq E(u_0) < E_{critical}$, where $E_{critical}$ is the energy of the ground state of $-\Delta Q + Q + \lambda Q^3 = 0$, a contradiction follows. Thus we conclude the above two cases as

Lemma 5.8. *For any sequence $t_n \to \infty$, up to a subsequence, there exists one equilibrium point Q to (1.1) such that*

$$u(t_n) = Q(x) + o_{H^1}(1). \quad \partial_t u(t_n) = o_{L^2}(1). \tag{5.34}$$

Step five. In this step, we accomplish the proof of Theorem 1.1 and part (II) of Theorem 1.2. From Lemma 5.8, there exists $t_n \to \infty$ and an equilibrium Q such that

$$u(t_n) \to Q \text{ in } H^1, \;\; \partial_t u \to 0, \text{ in } L^2. \tag{5.35}$$

If we can prove

$$\int_0^\infty \|u_t\|_{L^2} dt < \infty, \tag{5.36}$$

then $u(t)$ converges to Q in L^2. Since $\underset{t \geq t_0}{\cup} u(\cdot, t)$ is relatively compact in H^1 (by Lemma 5.8), we can conclude

$$u(t) \to Q \text{ in } H^1.$$

Therefore it suffices to prove (5.36).

For simplicity, we denote $E(u(t), \partial_t u(t))$ by $E(u)$ below. Define an auxiliary functional $H(t)$ as follows:

$$H(t) = E(u) - J(Q) - \varepsilon \big\langle \Delta u - u - b(x)u - \lambda u^3, u_t \big\rangle_{H^{-1}}.$$

Direct calculations show

$$
\begin{aligned}
H'(t) =& -2\alpha \left\|\partial_t u\right\|_{L^2}^2 + \varepsilon \left\{2\alpha\left\langle \Delta u - u - b(x)u - \lambda u^3, u_t\right\rangle_{H^{-1}}\right. \\
& \left. - \left\|\Delta u - u - b(x)u + u^3\right\|_{H^{-1}}^2 - \left\langle \Delta u_t - u_t - b(x)u_t + 3u^2 u_t, u_t\right\rangle_{H^{-1}}\right\}.
\end{aligned}
$$

It is easy to verify,

$$
\begin{aligned}
\left|\left\langle \Delta u_t - u_t - b(x)u_t - 3\lambda u^2 u_t, u_t\right\rangle_{H^{-1}}\right| &\lesssim \left\|u_t\right\|_{L^2}\left(\left\|u_t\right\|_{L^2} + \left\|u_t\right\|_{L^2}\left\|u\right\|_{H^1}^2\right) \\
\left|\left\langle \Delta u - u - b(x)u - \lambda u^3, u_t\right\rangle_{H^{-1}}\right| &\leq \left\|u_t\right\|_{L^2}\left\|\Delta u - u - b(x)u - \lambda u^3\right\|_{H^{-1}}.
\end{aligned}
$$

Therefore, we have

$$
H'(t) \leq -(2\alpha - C\varepsilon)\left\|\partial_t u\right\|_{L^2}^2 - \left\|\Delta u - u - b(x)u - \lambda u^3\right\|_{H^{-1}}^2. \tag{5.37}
$$

Choosing $\varepsilon > 0$ sufficiently small, we obtain $H'(t) \leq 0$. Since $H(t)$ is decreasing respect to t, and bounded below, by (5.35)

$$
\lim_{t\to\infty} H(t) = 0.
$$

If there exists a time t_1 such that $H(t_1) = 0$, then $H(t) \equiv 0$, for $t > t_1$, which implies by energy equality (5.37) that $u_t = 0$, for $t > t_1$. Thus $u(t) = Q$ for $t > t_1$, and we are done. Now we assume for $t > 0$

$$
H(t) > 0.
$$

For $\theta \in (0, \frac{1}{2})$, Lojasiewicz-Simon inequality implies that if $\|u - Q\|_{H^1} < \sigma$,

$$
\begin{aligned}
[H(t)]^{1-\theta} &\lesssim |J(u) - J(Q)|^{1-\theta} + \left\|u_t\right\|_{L^2}^{2(1-\theta)} + \left\|\Delta u - u - b(x)u - \lambda u^3\right\|_{H^{-1}} + \left\|u_t\right\|_{L^2}^{\frac{1-\theta}{\theta}} \\
&\lesssim \left\|u_t\right\|_{L^2} + \left\|\Delta u - u - b(x)u - \lambda u^3\right\|_{H^{-1}}.
\end{aligned}
$$

Thus we obtain from direct calculations that if $\|u - Q\|_{H^1} < \sigma$,

$$
-\frac{d}{dt}[H(t)]^{\theta} = -\theta H'(t)[H(t)]^{\theta-1} \gtrsim \theta\left(\left\|u_t\right\|_{L^2} + \left\|\Delta u - u - b(x)u - \lambda u^3\right\|_{H^{-1}}\right). \tag{5.38}
$$

By (5.35), for any $0 < \varepsilon \ll \sigma$, there is an integer N such that when $n \geq N$,

$$
\left\|u(t_n) - Q\right\|_{H^1} < \varepsilon, \quad [H(t_n)]^{\theta} < \varepsilon. \tag{5.39}
$$

For $n \geq N$, let

$$
\tilde{t}_n = \sup\left\{t \geq t_n : \left\|u(s) - Q\right\|_{H^1} < \sigma, \forall s \in [t_n, t]\right\}.
$$

If there exists a $n_0 \geq N$ such that $\tilde{t}_{n_0} = \infty$, then the definition of $\tilde{t}_{n_0}$ and (5.38) give

$$\int_{t_{n_0}}^{\infty} \|u_t\|_{L^2} dt \leq \sup_{t \geq t_{n_0}} [H(t)]^{\theta},$$

which is exactly (5.36), and we are done. Otherwise, we have $\tilde{t}_n < \infty$ for all $n > N$. Then the definition of $\tilde{t}_n$, (5.39), (5.38) yield

$$\begin{aligned} \left\|u(\tilde{t}_n) - Q\right\|_{L^2} &\lesssim \int_{t_n}^{\tilde{t}_n} \|u_t\|_{L^2} dt + \|u(t_n) - Q\|_{L^2} \\ &\lesssim [H(t_n)]^{\theta} + \|u(t_n) - Q\|_{L^2} \lesssim \varepsilon. \end{aligned}$$

Thus, as $n \to \infty$,

$$u(\tilde{t}_n) \to Q, \text{ in } L^2.$$

By the relative compactness of the orbit (by Lemma 5.8), up to a subsequence of $u(\tilde{t}_n)$, we have

$$u(\tilde{t}_n) \to Q, \text{ in } H^1.$$

Hence, there exists an integer $N' > N$, such that when $n > N'$,

$$\|u(\tilde{t}_n) - Q\|_{H^1} < \frac{\sigma}{2},$$

which contradicts with the definition of $\tilde{t}_n$. Thus Theorem 1.1 is proved.

6 Appendix A. Global existence implies the bounded-ness of the trajectory

In this appendix, we in addition assume $\sigma(-\Delta + b(x) + 1) = (0, \infty)$, $\lambda = -1$. We will prove if the solution is globally well-defined, then the trajectory is bounded. The proof of Lemma 6.1 and Lemma 6.2 is adapted from T. Cazenave [3] and N. Burq, G. Raugel, W. Schlag [2]. For completeness, we give the details.

Lemma 6.1. *For $b(x)$ in Theorem 1.2, $\lambda = -1$, any initial data $(u_0, u_1) \in \mathcal{H}$, then*
(i) either $u(x,t)$ blows up at finite time,
(ii) or $u(x,t)$ exists globally and for some $C > 0$ independent of $t \geq 0$

$$\sup_{t \in [0,\infty)} \|(u, \partial_t u)(t)\|_{\mathcal{H}} \leq C.$$

Proof. Assume that u is global in forward time, it suffices to prove $\|(u(t), \partial_t u(t))\|_{\mathcal{H}} \leq C$, for

some $C > 0$ independent of $t \geq 0$.

Step 1. Define $F(u, \partial_t u) = (4+\varepsilon) \int_{\mathbb{R}^3} |\partial_t u|^2 dx + \varepsilon \int_{\mathbb{R}^3} |\nabla u|^2 dx + \varepsilon \int_{\mathbb{R}^3} \left(|u|^2 + b(x)|u|^2\right) dx$, and $f(u) = \int_{\mathbb{R}^3} |u|^2 dx$, where ε is sufficiently small. Then immediate calculations show

$$f'' \geq F(u, \partial_t u) - 2(2+\varepsilon) E(u, u_t) - 4\alpha \int_{\mathbb{R}^3} u_t u dx. \tag{6.1}$$

Since $E(u, u_t)$ is decreasing, we conclude that

$$f'' \geq F(u, \partial_t u) - 2(2+\varepsilon) E(u_0, u_1) - 4\alpha \int_{\mathbb{R}^3} u_t u dx, \tag{6.2}$$

Hölder inequality and Young inequality imply

$$\left|f'(t)\right| \sqrt{\varepsilon(4+\varepsilon)} \leq \varepsilon \left\|u\right\|_2^2 + (4+\varepsilon) \left\|u_t\right\|_2^2,$$

which combined with $\langle -\Delta u + b(x)u, u\rangle \geq 0$, (6.2) gives

$$f'' \geq \sqrt{\varepsilon(4+\varepsilon)} \left(\left|f'(t)\right| - \frac{2(2+\varepsilon)}{\sqrt{\varepsilon(4+\varepsilon)}} E(u_0, u_1) \right) - 4\alpha \int u_t u dx. \tag{6.3}$$

Step 2. In Step 2, we prove if u is a global solution to (1.1), then

$$\frac{d}{dt} \left[(\varepsilon f(u(t)) - 2(2+\varepsilon) E(u_0, u_1))^+ \right] \leq 0 \tag{6.4}$$

$$f(u(t)) \leq \sup \left(f(u_0), \frac{2(2+\varepsilon)}{\varepsilon} E(u_0, u_1) \right). \tag{6.5}$$

Since (6.5) is a corollary of (6.4), it suffices to prove (6.4). Assume that $f(t) = f(u(t))$, $g(t) = f(t) - \frac{2+\varepsilon}{\varepsilon} E(u_0, u_1)$. If (6.4) fails, then there exists $t_1 > 0$ such that

$$g'(t_1) > 0, \ \ g(t_1) > 0.$$

Define $\mathcal{A} = \{t \geq t_1 : g'(t) \geq 0, g(t) \geq g(t_1)\}$. Obviously, $\mathcal{A}$ is a closed non-empty set, we claim for every $t_0 \in \mathcal{A}$, there exits a $\delta > 0$ depending on t_0 such that $(t_0, t_0 + \delta) \subset \mathcal{A}$. In fact, for fixed $t_0 \in \mathcal{A}$, we have $g(t_0) \geq g(t_1) > 0$, then there exists $\delta(t_0) > 0$ such that $g(t) > \frac{1}{2} g(t_0)$. For all $t > 0$, by (6.2) and $\langle Hu, u\rangle \geq 0$, we obtain

$$g'' + 2\alpha g' \geq \varepsilon g(t). \tag{6.6}$$

For $t \in (t_0, t_0 + \delta)$, Gronwall's inequality gives

$$e^{2\alpha t} g'(t) \geq e^{2\alpha t_0} g'(t_0) + \int_{t_0}^{t} \varepsilon e^{2\alpha} g(s) ds.$$

Since $g(t) > \frac{1}{2}g(t_0)$ on this interval, we conclude that

$$g'(t) > 0,$$

for $t \in (t_0, t_0+\delta)$. Thus $g(t)$ is increasing on this interval, which implies $g(t) > g(t_0) \geq g(t_1)$, namely $(t_0, t_0+\delta) \subset \mathcal{A}$. Next, we prove $\mathcal{A} = [t_1, \infty)$. Suppose that $t_* \in (t_1, \infty)$ and $t_* \notin \mathcal{A}$. Since $\mathcal{A}$ is closed, there exists $T \in \mathcal{A}$ such that $T = \max\{t \leq t_* : t \in \mathcal{A}\}$. By $(t_1, \infty)\backslash\mathcal{A}$ is open, we have $T < t_*$. However, for $T \in \mathcal{A}$, there exits $\delta(T) > 0$ such that $(T, T+\delta) \subset \mathcal{A}$, a contradiction with the maximum of T. Thus we have proved for all $t \in [t_1, \infty)$,

$$g'(t) \geq 0,\ g(t) \geq g(t_1),$$

which combined with (6.6) yields

$$g'(t) \geq \frac{\varepsilon}{2\alpha}(1-e^{-\alpha t})g(\frac{t}{2}). \tag{6.7}$$

Hence for t sufficiently large, $g'(t) \gtrsim g(t_1) > 0$, which leads to $g(t) \to \infty$, as $t \to \infty$. Again by (6.7), we deduce $g'(t) \to \infty$ as $t \to \infty$. We introduce an auxiliary functional $y(t)$ as follows,

$$y(t) = \frac{1}{2}\|u(t)\|_{L^2}^2 + \alpha \int_0^t \|u(s)\|_{L^2}^2 ds.$$

Then be the definition of g, we have

$$y(t) \to \infty, \quad y'(t) \to \infty, \quad \text{as } t \to \infty. \tag{6.8}$$

(6.13) in the following Lemma 6.2 gives

$$y(t)\ddot{y}(t) \geq c(\dot{y}(t))^2, \quad \text{for } t \gg 1.$$

It is easy to see this contradicts with (6.8), therefore, (6.4) holds.

Step 3. In Step 3, we prove for global solution $u(x,t)$, there exists $M(u_0, u_1) > 0$ such that

$$f'(t) \leq \frac{2(2+\varepsilon)}{\sqrt{\varepsilon(4+\varepsilon)}}E(u_0, u_1) \tag{6.9}$$

$$f'(t) \geq -M(u_0, u_1). \tag{6.10}$$

Let $h(t) = f'(t) - \big(2(2+\varepsilon)/\sqrt{\varepsilon(4+\varepsilon)}\big)E(u_0, u_1)$, then (6.3) gives

$$h'(t) \geq ah(t) - 4\alpha \int u u_t dx,$$

where $a=\sqrt{\varepsilon(4+\varepsilon)}$. By Gronwall's inequality and integration by parts, we have

$$
\begin{aligned}
h(t)-e^{a(t-t_2)}h(t_2) &\geq -2\alpha\int_{t_2}^{t} e^{-as}f'(s)ds \\
&= 2\alpha a\int_{t_2}^{t} e^{-as}f(s)ds - 2\alpha e^{-as}f(s)\big|_{t_2}^{t}.
\end{aligned}
$$

$f(t)$ is uniform bounded by Step 2, thus there exists some $M>0$, such that

$$h(t)\geq e^{a(t-t_2)}h(t_2)-M. \tag{6.11}$$

If there exists $t_2>0$ such that $h(t_2)>0$, then (6.11) yields $h(t)\to\infty$, namely $f'(t)\to\infty$. This contradicts with the bound of $f(t)$. Therefore, (6.9) has been proved. Let $k(t)=-f'(t)-\big(2(2+\varepsilon)/\sqrt{\varepsilon(4+\varepsilon)}\big)E(u_0,u_1)$, then

$$-k'(t)\geq ak(t)-4\alpha\int uu_t dx.$$

Similar arguments as the proof of (6.9) give

$$-f'(t)-\frac{2(2+\varepsilon)}{\sqrt{\varepsilon(4+\varepsilon)}}E(u_0,u_1)\leq \sup(0,k(0))+M_1,$$

for some $M_1>0$. Hence (6.10) follows. As a corollary of (6.2), (6.9), (6.10) and (6.5), we have

$$\int_t^{t+T} F(u(s),\partial_s u(s))ds \leq C(u_o,u_1)T. \tag{6.12}$$

Step 4 Define $s(t)=\frac{1}{2}\int|\nabla u|^2+|u|^2+b(x)|u|^2+|u_t|^2dx$. Then (1.1) shows

$$s'(t)=\int u^3u_t dx-2\alpha\int|u_t|^2dx.$$

Sobolev embedding implies

$$s'(t)\leq \|u_t\|_{L^2}\|u\|_{H^1}^3\leq Cs(t)\|u\|_{H^1}^2.$$

We deduce from Gronwall's inequality that

$$s(t+\tau)\leq s(t)\exp\left\{\int_t^{t+\tau} C\left\|u(s)\right\|_{H^1}^2 ds\right\}.$$

For $t\geq 1$, $0\leq\tau\leq 1$, we have

$$s(t)\leq s(t-\tau)\exp\left\{\int_{t-1}^{t} C\left\|u(s)\right\|_{H^1}^2 ds\right\}.$$

Integrating the above inequality for $\tau \in [0,1]$, we obtain

$$s(t) \leq \left(\int_{t-1}^{t} s(\tau) d\tau \right) \exp \left\{ \int_{t-1}^{t} C \|u(s)\|_{H^1}^2 \, ds \right\}.$$

Since $s(t) \lesssim F(u(t), \partial_t u(t))$, $\|u\|_{H^1}^2 \lesssim F(u(t), \partial_t u(t))$, (6.12) yields our lemma for $t \geq 1$. When $t \in [0,1]$, the bound of $\|(u, \partial_t u)\|_{\mathcal{H}}$ is obvious by the blow-up criterion. $\square$

Lemma 6.2. *If $E(u_0, u_1) < 0$, then the solution to (1.1) blows up at finite time.*

Proof. Define $K(u) = \int |\nabla u|^2 + |u|^2 + b(x)|u|^2 - |u|^4 dx$, $\tilde{H} u = -\Delta u + u + b(x) u$. Then we have

$$E(u) - \frac{1}{4} K(u) = \int \frac{1}{4} |\nabla u|^2 + \frac{1}{4} |u|^2 + \frac{1}{2} |\partial_t u|^2 + \frac{1}{4} b(x) |u|^2 dx.$$

Hence,

$$E(u) \geq \frac{1}{4} K(u) + \frac{1}{4} \left\langle \tilde{H} u, u \right\rangle \geq \frac{1}{4} K(u).$$

If $E(u_0) < -\delta$, since the energy is decreasing, we have $K(u) < -4\delta$. Consider

$$y(t) = \frac{1}{2} \|u(t)\|_{L^2}^2 + \alpha \int_0^t \|u(s)\|_{L^2}^2 ds.$$

Then

$$\begin{aligned} \dot{y}(t) &= \langle \dot{u}(t), u(t) \rangle + \alpha \|u(t)\|_{L^2}^2 \\ &= \langle \dot{u}(t), u(t) \rangle + \alpha \|u(0)\|_{L^2}^2 + 2\alpha \int_0^t \langle \dot{u}(s), u(s) \rangle ds. \end{aligned}$$

And

$$\begin{aligned} \ddot{y}(t) &= \langle \ddot{u}(t), u(t) \rangle + \langle \dot{u}(t), \dot{u}(t) \rangle + 2\alpha \langle \dot{u}(t), u(t) \rangle \\ &= \|u(t)\|_{L^2}^2 + \langle \ddot{u}(t) + 2\alpha \dot{u}(t), u(t) \rangle \\ &= \|u(t)\|_{L^2}^2 - K(u) \geq 4\delta. \end{aligned}$$

Therefore, we have $\dot{y}(t) \to \infty$, and $y(t) \to \infty$, as $t \to \infty$. Notice that

$$\begin{aligned} \ddot{y}(t) &= \|u(t)\|_{L^2}^2 - K(u) \\ &= 3 \|\dot{u}(t)\|_{L^2}^2 + \langle H u, u \rangle - 4E(u(t)), \end{aligned}$$

and

$$E(u(t)) = E(u(0)) + \int_0^t \dot{E}(s) ds$$

$$= E(u(0)) - 2\alpha \int_0^t \|\dot{u}(s)\|_{L^2}^2 \, ds.$$

Thus

$$\ddot{y}(t) \geq 3 \|\dot{u}(t)\|_{L^2}^2 + \left\langle \tilde{H}u, u \right\rangle - 4E(u(0)) + 8\alpha \int_0^t \|\dot{u}(s)\|_{L^2}^2 \, ds.$$

Now we calculate

$$\begin{aligned}
&y(t)\ddot{y}(t) - c(\dot{y}(t))^2 \\
&\geq \left(\frac{1}{2}\|u(t)\|_{L^2}^2 + \alpha \int_0^t \|\dot{u}(s)\|_{L^2}^2 \, ds\right)\left(3\|\dot{u}(t)\|_{L^2}^2 + \langle Hu, u\rangle - 4E(u(0)) + 8\alpha \int_0^t \|\dot{u}(s)\|_{L^2}^2 \, ds\right) \\
&- c\left[\|u(t)\|_{L^2}\|\dot{u}\|_{L^2} + 2\alpha\left(\int_0^t \|u(s)\|_{L^2}^2 \, ds\right)^{1/2}\left(\int_0^t \|\dot{u}(s)\|_{L^2}^2 \, ds\right)^{1/2} + \alpha\|u(0)\|_{L^2}^2\right]^2.
\end{aligned}$$

Young inequality implies

$$\begin{aligned}
&c\left[\|u(t)\|_{L^2}\|\dot{u}\|_{L^2} + 2\alpha\left(\int_0^t \|u(s)\|_{L^2}^2 \, ds\right)^{1/2}\left(\int_0^t \|\dot{u}(s)\|_{L^2}^2 \, ds\right)^{1/2} + \alpha\|u(0)\|_{L^2}^2\right]^2 \\
&\leq c\,(1+\varepsilon)\left(\frac{1}{2}\|u(t)\|_{L^2}^2 + \alpha \int_0^t \|\dot{u}(s)\|_{L^2}^2 \, ds\right)\left(2\|\dot{u}(t)\|_{L^2}^2 + 4\alpha \int_0^t \|\dot{u}(s)\|_{L^2}^2 \, ds\right) \\
&+ c\left(1 + \frac{1}{\varepsilon}\right)\alpha^2 \|u(0)\|_{L^2}^4.
\end{aligned}$$

Let $b = c(1+\varepsilon)$, $C = c\alpha^2(1+\frac{1}{\varepsilon})\|u(0)\|_{L^2}^2$, the right side of the above inequality can be bound by

$$\leq y(t)\left(2b\|\dot{u}(t)\|_{L^2}^2 + 4b\alpha \int_0^t \|\dot{u}(s)\|_{L^2}^2 \, ds\right) + C.$$

Thus, we have

$$\begin{aligned}
&y(t)\ddot{y}(t) - c(\dot{y}(t))^2 \\
&\geq y(t)\left((3-2b)\|\dot{u}(t)\|_{L^2}^2 + \left\langle \tilde{H}u, u \right\rangle - 4E(u(0)) + 4\alpha(2-b)\int_0^t \|\dot{u}(s)\|_{L^2}^2 \, ds\right) - C \\
&\equiv y(t)\Psi(t) - C.
\end{aligned}$$

Suppose that $\varepsilon > 0$ is sufficiently small, $c = 1^+$, then

$$\Psi(t) \geq \eta\dot{y}(t) - 4E(0) + q(t),$$

where $q(t)$ is nonnegative. Hence, we conclude

$$y(t)\ddot{y}(t) - c(\dot{y}(t))^2 \geq y(t)\left(\eta\dot{y}(t) - 4E(0) + q(t)\right) - C.$$

Since $\dot{y}(t), y(t) \to \infty$, as $t \to \infty$, we get

$$y\dot{y}(t) \geq c(\dot{y}(t))^2, \tag{6.13}$$

for t sufficiently large. This contradicts with $y(t) \to \infty$. $\square$

7 Appendix B. Notions of Gradient systems and some examples

In this appendix, we recall the definition of gradient systems and add some remarks. First we recall the definition of Lyapunov functions.

Definition 7.1. *(Lyapunov function) Suppose that H is a complex metric space and $S(t)$ is a nonlinear C_0-semigroup defined on H. A continuous function $V : H \to \mathbb{R}$ is called a Lyapunov function with respect to $S(t)$ if the following two conditions are satisfied:*
(i) For any $x \in H$, $V(S(t)x)$ is monotone non-increasing in t;
(ii) $V(x)$ is bounded from below.

The gradient system is defined as follows.

Definition 7.2. *(Gradient System) Suppose that V is a Lyapunov function. Then $(H, S(t), V)$ is called a gradient system if*
(i) for any $x \in H$, there exists $t_0 > 0$ such that $\bigcup_{t \geq t_0} S(t)x$ is relatively compact in H.
(ii) if for $t > 0$, $V(S(t)x) = V(x)$, then x is a fixed point of the semigroup $S(t)$.

Many models are gradient systems, such as nonlinear reaction-diffusion equations (see for instance H. Matano [24]), Cahn-Hilliard equations (see for instance P. Rybka, K.H. Hoffnlann [33]), heat flows of harmonic maps, Landau-Lifshitz-Gilbert equations. Generally, the global solution with bounded trajectory to gradient systems defined in bounded domains will converge to an equilibrium. But there do exit some gradient systems for which some bounded solution does not converge, see for instance P. Polaik, F. Simondon [32] and P. Polaik, K.P. Rybakowski [31] for semilinear heat equations.

References

[1] H. Berestycki, P.L. Lions. *Nonlinear scalar field equations, II existence of infinitely many solutions.* Archive for Rational Mechanics and Analysis, 1983, 82: 347-375.

[2] N. Burq, G. Raugel, W. Schlag, *Long time dynamics for damped Klein-Gordon equations*, arXiv:1505.05981.

[3] T. Cazenave. *Uniform estimates for solutions of nonlinear Klein-Gordon equations*. Journal of functional analysis, 1985, 60: 36-55.

[4] R.Cote, C. Kenig, A. Lawrie, W, Schlag. *Profiles for the radial focusing 4d energy-critical wave equation*. arXiv preprint arXiv:1402.2307, 2014.

[5] R. Cote, C. Kenig, A. Lawrie, W. Schlag. *Characterization of large energy solutions of the equivariant wave map problem*: II. American Journal of Mathematics, 2015, 137: 209-250.

[6] R. Cote, C. Kenig, A. Lawrie, W. Schlag. *Characterization of large energy solutions of the equivariant wave map problem*: I. American Journal of Mathematics, 2015, 137: 139-207.

[7] T. Duyckaerts, C. Kenig, F. Merle, *Classification of radial solutions of the focusing, energy-critical wave equation*, Cambridge Journal of Mathematics, 2013, 1: 75-144.

[8] T. Duyckaerts, C. Kenig, F. Merle. *Universality of blow-up profile for small radial type II blow-up solutions of the energy-critical wave equation*. Journal of the European Mathematical Society, 2011, 13: 533-599.

[9] T. Duyckaerts, C. Kenig, F. Merle. *Universality of the blow-up profile for small type II blow-up solutions of the energy-critical wave equation: the nonradial case*. Journal of the European Mathematical Society, 2012, 14: 1389-1454.

[10] E. Feireisl, *Attractors for semilinear damped wave equations on* R^3, Nonlinear Analysis: Theory, Methods and Applications, 1994, 23, 187-195.

[11] E. Feireisl, *Global attractors for semilinear damped wave equations with supercritical exponent*. Journal of differential equations, 1995, 116: 431-447.

[12] E. Feireisl, *Finite energy travelling waves for nonlinear damped wave equations*, Quarterly Journal of Applied mathematics LVI(1998), 55-70.

[13] J.K. Hale, G. Raugel. *Convergence in gradient-like systems with applications to PDE*. Zeitschrift für Angewandte Mathematik und Physik (ZAMP), 1992, 43: 63-124.

[14] A. Haraux, M. A. Jendoubi, *Convergence of bounded weak solutions of the wave equation with dissipation and analytic nonlinearity*, Calculus of Variations and Partial Differential Equations, 1999, 9, 95-124.

[15] A. Haraux, M. A. Jendoubi. *Convergence of Solutions of Second-Order Gradient-Like Systems with Analytic Nonlinearities*, Journal of Differential Equations, 1998, 144, 313-320.

[16] S. Ibrahim, N. Masmoudi, K. Nakanishi. *Scattering threshold for the focusing nonlinear KleinCGordon equation*, Analysis and PDE, 2011, 4, 3405-460.

[17] H. Jia, B.P. Liu, G.X. Xu. *Long time dynamics of defocusing energy critical 3+ 1 dimensional wave equation with potential in the radial case*. Communications in Mathematical Physics, 2015, 339: 353-384.

[18] H. Jia, C. Kenig, *Asymptotic decomposition for semilinear wave and equivariant wave map equations*, arXiv. 1503.06715.

[19] C. Kenig, A., Lawrie, W., Schlag. *Relaxation of wave maps exterior to a ball to harmonic maps for all data*. Geometric and Functional Analysis, 2014, 24: 610-647.

[20] C. Kenig, A. Lawrie, B.P. Liu, W. Schlag. *Stable soliton resolution for exterior wave maps in all equivariance classes*. Advances in Mathematics, 2015, 285: 235-300.

[21] Y.S. Kivshar, B.A. Malomed. *Dynamics of solitons in nearly integrable systems*. Reviews of Modern Physics, 1989, 61(4): 763.

[22] P.L. Lions. *The concentration-compactness principle in the calculus of variations. The locally compact case, part 2.* Annales de l'IHP Analyse non lineaire. 1984, 1: 223-283.

[23] Z. Li, L. Zhao. *Asymptotic decomposition for nonlinear damped Klein-Gordon equations*. arXiv:1511.00437.

[24] H. Matano, *Convergence of solutions of one-dimensional semilinear parabolic equations*, J. Math. Kyoto Univ., 18-2 (1978), 221-227.

[25] D.W. McLaughlin, A.C. Scott, *Perturbation analysis of fluxon dynamics*, Phys. Rev. A18 (1978) 1652.

[26] K. Nakanishi, W. Schlag, *Invariant manifolds and dispersive Hamiltonian Evolution Equations*, Zurich Lectures in Advanced mathematicas. European Mathematical Society (EMS), Zürich, 2011.

[27] K. Nakanishi, W. Schlag, *Global dynamics above the ground state energy for the focusing nonlinear Klein-Gordon equation*, J. Differetnial Equations, 250 (2011), 2299-2333.

[28] K. Nakanishi, W. Schlag, *Global dynamics above the ground state energy for the nonlinear Klein-Gordon equation without a radial assumption*, Arch. Rational Mech. Anal. 203 (2012) 809-851.

[29] P. J. Rabiera, C. A. Stuartb, *Exponential Decay of the Solutions of Quasilinear Second-Order Equations and Pohozaev Identities*, Journal of Differential Equations, 165 (2000), 199-234.

[30] M. Prizzi, K.P. Rybakowski. *Attractors for semilinear damped wave equations on arbitrary unbounded domains*. Topological Methods in Nonlinear Analysis, 2008, 31(1), 49-82.

[31] P. Polaik, K.P. Rybakowski, *Nonconvergent bounded trajectories in semilinear heat equations*. Journal of differential equations, 1996, 124, 472-494.

[32] P. Polaik, F. Simondon, *Nonconvergent bounded solutions of semilinear heat equations on arbitrary domains*. Journal of Differential Equations, 2002, 186, 586-610.

[33] P. Rybka, K.H. Hoffnlann K H. *Convergence of solutions to Cahn-Hilliard equation*. Communications in partial differential equations, 24 (1999), 1055-1077.

[34] A. Soffer, *Soliton dynamics and scattering*, International Congress of Mathematicians. Vol. III. 459-471. European Mathematical Society , Zurich, 2006.

[35] A. Soffer, M. I. Weinstein, *Resonances, radiation damping and instabilitym in Hamiltonian nonlinear wave equations*, Invent. Math. 136 (1999), 9-74.

[36] L. Simon, *Asymptotics for a Class of Non-Linear Evolution Equations, with Applications to Geometric Problems*, Annals of Math. 118 (1983), 525-571.

[37] T. Tao, *A (concentration-) compact attractor for highdimensional nonlinear Schrödinger equations*, Dynamics of Partial Differential Equations, 4 (2007) 1-53.

[38] T. Tao, *A Global Compact Attractor for High-Dimensional Defocusing Non-linear Schr?dinger Equations with Potential*. Dynamics of PDE, 2008, 5: 101-116.

[39] K. Yajima, *The W k, p-continuity of wave operators for Schrödinger operators*, Journal of the Mathematical Society of Japan, 47 (1995), 551-581.

[40] F. Zhang, Y.S. Kivshar, B.A. Malomed, L. Vazquez, *Kink capture by a local impurity in the sine Gordon model*, Phys. Lett. A159 (1991) 318-322.